# CONCEPT QUICKSTART – Methods of Solving First Order, First Degree Differential Equations

**Unit:** Unit 9: Differential Equations

**Subject: For CBSE Class 12 Mathematics**

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## SECTION 1: UNDERSTANDING THE CONCEPT

This section lays the essential foundation for mastering first-order, first-degree differential equations. Before you can solve a problem, you must understand the language of the problem. A firm grasp of the 'what', 'why', and core vocabulary is the first and most critical step toward developing an effective problem-solving strategy.

### 1.1 What Is This Topic?

This topic introduces systematic methods for finding a function, y = f(x), when you are given an equation that involves its first derivative, dy/dx. Essentially, these methods provide a clear roadmap to work backward—from a mathematical description of a rate of change to the original function that describes the relationship between the variables.

A common point of confusion is to think there is a single, universal formula for solving all first-order differential equations. This is incorrect. The methods discussed here—specifically **Variables Separable** and **Homogeneous Equations**—are distinct techniques, each designed for a specific *type* of equation. Learning to identify the structure of an equation is the key to choosing the correct method.

### 1.2 Why It Matters

Differential equations are the fundamental mathematical language used to model real-world phenomena involving rates of change. Their importance extends far beyond the classroom, forming the bedrock of modern scientific investigation.

As highlighted in the NCERT curriculum, these equations are applied across numerous fields, including:

- **Physics:** Modeling motion, heat transfer, and electrical circuits.
- **Chemistry:** Describing rates of chemical reactions.
- **Biology:** Analyzing population growth and the spread of diseases.
- **Anthropology** and **Geology:** Modeling population dynamics and geological processes.
- **Economics:** Modeling financial growth and market trends.

Mastering the techniques to solve these equations allows us to analyze, understand, and predict the behavior of complex systems over time.

### 1.3 Prior Learning Connection

Your ability to solve differential equations directly depends on your fluency with a few key concepts from earlier studies.

- **Differentiation:** A differential equation is, by definition, an equation containing derivatives. You must be comfortable with the concept of a derivative (dy/dx) to understand the structure of the problems you are asked to solve.
- **Integral Calculus:** This is the core operational tool. Solving a differential equation is fundamentally an act of 'undoing' a derivative to find the original function, and integration is the mathematical process for doing exactly that.

### 1.4 Core Definitions

The following terms are the building blocks of this entire chapter. Understanding them precisely is non-negotiable.

- **Differential Equation**
    - **NCERT Reference:** Section 9.2, p. 301
    - **Definition:** An equation involving the derivative (or derivatives) of a dependent variable with respect to an independent variable (or variables).
    - **Used In:** All problem types.
- **Order of a Differential Equation**
    - **NCERT Reference:** Section 9.2.1, p. 301
    - **Definition:** The order of the highest order derivative of the dependent variable involved in the differential equation.
    - **Used In:** Classifying all differential equations. This topic focuses on first-order equations.
- **Degree of a Differential Equation**
    - **NCERT Reference:** Section 9.2.2, p. 302
    - **Definition:** The highest power (positive integral index) of the highest order derivative, provided the differential equation is a polynomial equation in its derivatives.
    - **Used In:** Classifying all differential equations. This topic focuses on first-degree equations.

- **General Solution**
  - **NCERT Reference:** Section 9.3, p. 305
  - **Definition:** A solution which contains arbitrary constants (also called a primitive). It represents a family of solution curves.
  - **Used In:** All problem types where initial conditions are not given.
- **Particular Solution**
  - **NCERT Reference:** Section 9.3, p. 305
  - **Definition:** A solution obtained from the general solution by giving particular values to the arbitrary constants. It is free from arbitrary constants.
  - **Used In:** All problem types where initial conditions (e.g., "y=1 when x=0") are provided.
- **Differential Equation with Variables Separable**
  - **NCERT Reference:** Section 9.4.1, p. 307
  - **Definition:** A first-order, first-degree differential equation of the form dy/dx = F(x, y) that can be expressed as dy/dx = g(x)h(y).
  - **Used In:** Problem Type: Variables Separable.
- **Homogeneous Function**
  - **NCERT Reference:** Section 9.4.2, p. 312
  - **Definition:** A function F(x, y) is said to be a homogeneous function of degree n if $F(\lambda x, \lambda y) = \lambda^n F(x, y)$ for any nonzero constant λ.
  - **Used In:** Problem Type: Homogeneous Differential Equations.
- **Homogeneous Differential Equation**
  - **NCERT Reference:** Section 9.4.2, p. 313
  - **Definition:** A differential equation of the form dy/dx = F(x, y) where F(x, y) is a homogeneous function of degree zero.
  - **Used In:** Problem Type: Homogeneous Differential Equations.

With this foundational vocabulary established, we can now examine how the NCERT textbook presents and applies these concepts.

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## SECTION 2: WHAT NCERT SAYS

This section distills the core principles, strategies, and examples directly from the NCERT textbook. It highlights the most important statements and showcases representative problems to build a clear picture of the textbook's approach to solving first-order, first-degree differential equations.

### 2.1 Key Statements

The NCERT textbook establishes several fundamental principles that guide the entire problem-solving process.

1. **Condition for Degree:** The 'degree' of a differential equation can only be determined if the equation is a polynomial in its derivatives (like y', y'', etc.). If derivatives appear inside other functions, such as sin(y'), the degree is not defined (p. 302).
2. **Nature of a Solution:** Unlike algebraic equations, which are solved for a numerical value, a differential equation is solved for a function, $y = \phi(x)$. This function is the solution (p. 304).
3. **Arbitrary Constants:** The **general solution** contains arbitrary constants (like C) and represents an entire family of curves. A **particular solution** is found by using given conditions to determine the value of these constants, resulting in a single, specific curve (p. 305).
4. **Variable Separation Method:** The central technique for this method is to algebraically rearrange the equation so that all y terms are grouped with dy and all x terms are grouped with dx. Once separated, each side is integrated (p. 307).
5. **Homogeneous Equation Test:** A differential equation dy/dx = F(x, y) is homogeneous if the function F(x, y) can be expressed solely in terms of the ratio y/x (p. 313).
6. **Homogeneous Solution Strategy:** The standard approach for solving a homogeneous differential equation is to use the substitution y = vx. This substitution cleverly transforms the equation into a new one where the variables v and x can be separated (p. 313).

### 2.2 Examples and Exercises

The worked examples in the textbook are designed to illustrate these principles in action.

- **Example 4 (p. 307):** This demonstrates the most straightforward application of the **variable separation** method to find a general solution. It is important because it establishes the fundamental technique of separating and integrating.
- **Example 6 (p. 308):** This builds on the previous example by showing how to find a ***particular solution***. It uses the variable separation method first and then applies the

given initial condition (y=1 when x=0) to calculate the specific value of the integration constant C.

- **Example 10 (p. 314):** This provides a complete walkthrough for a **homogeneous equation**. It shows the full process: (1) proving the equation is homogeneous, (2) applying the y = vx substitution, (3) solving the new separable equation, and (4) substituting back to get the final general solution.
- **Example 13 (p. 319):** This example makes a crucial connection between a geometric concept and a differential equation. It translates the phrase "slope of the tangent" into its mathematical equivalent, dy/dx, and then solves the resulting homogeneous equation to find the equation for a family of curves.

For practice, the textbook provides the following exercise sets:

- **Variables Separable:** Exercise 9.3 (Questions 1-23)
- **Homogeneous Equations:** Exercise 9.4 (Questions 1-17)

Notice the progression: NCERT starts with the most basic general solution (Ex. 4), adds a layer of complexity with initial conditions (Ex. 6), then introduces an entirely new method for a different class of problem (Ex. 10), and finally connects the method to a real-world geometric application (Ex. 13).

Understanding the textbook's presentation is the first step. The next section will organize this knowledge into a structured, actionable guide for solving problems yourself.

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**SECTION 3: PROBLEM-SOLVING AND MEMORY**

This section is a strategic guide designed for practical application. We will move beyond theory and focus on execution. Here, we break down the methods into recognizable types, provide step-by-step instructions, and highlight common pitfalls to build a reliable toolkit for exams.

**3.1 Problem Types**

Your first task when facing a differential equation is to correctly identify its type.

- **Problem Type 1: Variables Separable**
  - **Structural Goal:** To algebraically manipulate the equation into the form h(y)dy = g(x)dx.
  - **Recognition Cues:**
    - **Surface:** The equation is given in the form dy/dx = F(x, y).

- **Structural:** The function F(x, y) can be factored into a product of a function of x only and a function of y only, i.e., F(x, y) = g(x)h(y).

- **What You're Really Doing:** You are isolating all x-related terms on one side of the equation and all y-related terms on the other. Then, you integrate each side independently to discover the underlying relationship between y and x.
- **NCERT References:** Examples 4–9 | Exercises 9.3, Q1–23.
- **Confusable Types:** Equations that look simple but cannot be separated, such as dy/dx = x + y. This equation fails the structural test because x + y cannot be factored into g(x)h(y). Always check for separability first.

- **Problem Type 2: Homogeneous**
  - **Structural Goal:** To transform the equation using the substitution y = vx into a new differential equation in terms of v and x that is solvable by the variables separable method.
  - **Recognition Cues:**
    - **Surface:** The equation dy/dx = F(x, y) often contains rational expressions where the sum of the powers of x and y in each term of the numerator is the same as in each term of the denominator (e.g., $(x^2 + y^2)/(xy)$).
    - **Structural:** The function F(x, y) is a homogeneous function of degree zero. This means it can be written entirely as a function of the ratio y/x, i.e., F(x, y) = g(y/x).
  - **What You're Really Doing:** You are using a standard substitution as a tool to convert a complex, non-separable equation into a simpler, separable one that you already know how to solve. The substitution y = vx is effective because it reduces a function of two variables (x and y) into a function of a single new variable (v = y/x). This reduction is what allows the new equation to become separable.
  - **NCERT References:** Examples 10–13 | Exercises 9.4, Q1–17.
  - **Confusable Types:** Non-homogeneous equations that might appear structurally similar but fail the formal test for homogeneity (F(λx, λy) ≠ F(x, y)). Always perform the test if you are unsure.

### 3.2 Step-by-Step Methods

- **Type: Variables Separable: Solution Method**
  - **Pre-Check:** Verify that the equation dy/dx = F(x, y) can be rewritten as dy/dx = g(x)h(y). If not, this method is not applicable.

- **Core Steps:**
  1. **Separate:** Rearrange the equation to group all y terms with dy on one side and all x terms with dx on the other. The standard form is (1/h(y))dy = g(x)dx.
  2. **Integrate:** Integrate both sides of the separated equation: ∫(1/h(y))dy = ∫g(x)dx.
  3. **Add Constant:** Add a single constant of integration, C, to one side of the resulting equation.
  4. **Solve for Particular Solution:** If initial conditions (e.g., a point ($x_0$, $y_0$)) are given, substitute these values into the general solution to find the value of C. Rewrite the equation with this value of C.

- **Type: Homogeneous: Solution Method**
  - **Pre-Check:** Verify that F(x, y) is a homogeneous function of degree zero. This can be done by showing that F(λx, λy) = F(x, y) or by rewriting the function entirely in terms of the ratio y/x.
  - **Core Steps:**
    1. **Setup:** Make the substitution y = vx. From this, differentiate with respect to x to get dy/dx = v + x(dv/dx).
    2. **Substitute & Simplify:** Replace all instances of y and dy/dx in the original equation with their v and x equivalents. The equation should simplify such that the x terms cancel out, leaving a new equation involving only v and x.
    3. **Apply Separable Method:** The new equation will be solvable using the variables separable method. Rearrange it to separate v and x.
    4. **Integrate:** Integrate both sides of the separated equation and add the constant of integration, C.
    5. **Back-Substitute:** Replace v with its original definition, y/x, to express the final solution in terms of the original variables, x and y.
  - **Variants:**
    - **Finding a Particular Solution:** For both types, if initial values are provided, use them in the final step to find the constant C.
    - **Alternative Homogeneous Form:** If the equation is given as dx/dy = F(x, y), use the substitution x = vy and proceed with a similar method (p. 314).

- **When NOT to Use:**
  - **Variables Separable:** When F(x, y) cannot be factored into g(x)h(y).
  - **Homogeneous:** When F(x, y) is not a homogeneous function of degree zero.

### 3.3 How to Write Answers

Presenting your solution clearly is as important as finding it. Follow this template for a logical and easy-to-grade answer.

- **Answer Template:** General Solution Framework
- **When to Use:** For any exam question asking for the general or particular solution of a first-order, first-degree differential equation.
- **Line-by-Line:**
  - **L1 (Given):** Start by writing the given differential equation.
  - **L2 (Identification):** State the type of equation and why. (e.g., "This is a homogeneous differential equation because F(x, y) is a homogeneous function of degree zero.")
  - **L3 (Method Setup):** For Variables Separable, show the equation rearranged with variables separated. For Homogeneous, state the substitution: "Let y = vx, which implies dy/dx = v + x(dv/dx)".
  - **L4 (Execution):** Show the key steps of substitution, simplification, and integration. Work neatly.
  - **L5 (General Solution):** State the resulting equation containing the constant C. Box this answer if it is the final result required.
  - **L6 (Particular Solution Step):** If required, write "Substituting the given condition x=..., y=..." and show the calculation to find C.
  - **L7 (Final Answer):** Write the final particular solution with the calculated value of C.
- **Essential Phrases:** "Separating the variables, we get...", "Integrating both sides...", "Substituting y = vx...", "Replacing v with y/x, the general solution is...".
- **General Rules:**

1. Always add the constant of integration C immediately after the integration step.

2. For homogeneous problems, clearly show both the initial substitution (y=vx) and the final back-substitution (v=y/x).

3. Simplify the final algebraic expression as much as is practical.

### 3.4 Common Mistakes

- **Common Pitfalls**
  - **Pitfall #1: Forgetting the Constant of Integration**
    - **Category:** Logic
    - **Occurs In:** Variables Separable & Homogeneous (Integration Step)
    - **Wrong:** $\int 2y\, dy = \int 2x\, dx$ leads to $y^2 = x^2$.
    - ✓ **Fix:** $\int 2y\, dy = \int 2x\, dx$ must lead to $y^2 = x^2 + C$. The constant C is essential for the *general* solution. Without C, you are providing only one particular solution (the one passing through the origin), not the entire family of functions that the general solution represents.
  - **Pitfall #2: Incorrect Separation of Variables**
    - **Category:** Algebra
    - **Occurs In:** Variables Separable (Setup Step)
    - **Wrong:** Trying to separate dy/dx = x + y as dy - y = x dx. This is algebraically invalid.
    - ✓ **Fix:** Recognize that terms like x + y cannot be factored into g(x)h(y). Misapplying the method here is an immediate sign of a fundamental misunderstanding. If separation isn't possible, you must move on to test for other types, like homogeneity.
  - **Pitfall #3: Errors in Back-Substitution**
    - **Category:** Algebra
    - **Occurs In:** Homogeneous (Final Step)
    - **Wrong:** After finding a solution in terms of v, such as log|v| = log|x| + C, forgetting to replace v with y/x.
    - ✓ **Fix:** The final answer must be in terms of the original variables. Always complete the last step: substitute v = y/x back into the equation.
- **Key Rules**
  - **Condition #1: Polynomial in Derivatives**
    - **Rule:** The concept of 'degree' is only defined if the differential equation can be written as a polynomial in its derivatives (y', y'', etc.).

- **When:** This is important when asked to classify an equation. For example, in an equation like sin(dy/dx) + y = 0, the order is 1, but the degree is not defined.
- **Linked:** Basic classification of DEs (p. 302).

### 3.5 Exam Strategy

- **Example Range:** Focus your revision on the methods demonstrated in the NCERT worked **Examples 4 through 13 (pp. 307-320)**. These cover the core techniques and applications.
- **Exercise Sets:** Gain proficiency by practicing a wide range of problems from **Exercise 9.3** (Variables Separable) and **Exercise 9.4** (Homogeneous).
- **Question Patterns:** Be prepared for three main types of questions:
  1. **Direct Solution:** "Find the general solution of the differential equation..."
  2. **Initial Value Problem:** "Find the particular solution... satisfying the condition y(...) = ...".
  3. **Geometric/Applied Problem:** "Find the equation of the curve given the slope of its tangent..." or a word problem involving rates.
- **Approach:** First, achieve complete mastery of the **Variables Separable** method. It is the simpler of the two and is also a mandatory sub-step for solving homogeneous equations. Second, practice **recognizing** a homogeneous equation quickly and executing the y=vx substitution procedure without error.

### 3.6 Topic Connections

This unit is not isolated; it builds on previous knowledge and provides tools for future studies.

- **Prerequisites:**
  - **Differentiation:** Required to understand the very definition and structure of a differential equation.
  - **Integration:** The primary mechanical skill used to solve for the unknown function.
- **Forward Links:**
  - **Physics:** Solving these equations is essential for modeling phenomena like radioactive decay, Newton's law of cooling, and simple harmonic motion.
  - **Biology:** Used to model population dynamics and growth rates.

- **Economics:** Applied to problems of continuous compounding interest and economic growth models (as seen in NCERT Example 9).

**3.7 Revision Summary**

- A first-order, first-degree differential equation involves dy/dx raised to the power of 1.
- The first method to always check is **Variables Separable**. This applies if dy/dx can be expressed as a product g(x)h(y).
- If variables are not separable, check if the equation is **Homogeneous**. This is true if dy/dx can be written as a function of the ratio y/x.
- The standard technique for solving homogeneous equations is the substitution **y = vx**, which transforms the problem into a variables separable type.
- A **general solution** represents a family of curves and must always include an arbitrary constant of integration, C.
- A **particular solution** represents a single, specific curve and is found by using given initial conditions to calculate the value of C.
- Never forget to add + C immediately after performing an integration.
- For homogeneous problems, always remember to perform the final **back-substitution** step, replacing v with y/x.