Concept QuickStart– Evaluation of Definite Integrals by Substitution

Unit 7: Integrals

Subject: For CBSE Class 12 Mathematics

SECTION 1: UNDERSTANDING THE CONCEPT

Integration by substitution is a pivotal technique in integral calculus, serving as one of the primary strategies for solving integrals that are not in a standard, immediately solvable form. Many complex functions cannot be integrated by simple inspection or direct application of basic formulas. The substitution method provides a systematic process to transform these challenging integrals into simpler, more recognizable forms by changing the variable of integration. This method is, in essence, the reverse of the chain rule for differentiation, providing a powerful way to undo the integration of composite functions.

1.1 What Is Integration by Substitution?

At its core, integration by substitution is the process of simplifying an integral by changing the independent variable from, for example, x to a new variable t. The goal is to select a substitution that transforms the original integrand into a function that can be solved using standard integration formulas.

A common misunderstanding is that any part of a function can be chosen for substitution. The key to a successful substitution, however, is more specific: one must choose a function within the integrand, let's call it g(x), whose derivative, g'(x), is also present as a factor in the original integrand. This relationship between a function and its derivative is what allows for a clean transformation of the entire integral into the new variable.

1.2 Why It Matters

Mastering integration by substitution is fundamental because it significantly broadens the range of functions that can be integrated. While basic integration formulas cover elementary functions, most real-world problems in science, engineering, and economics involve more complex expressions. This method is the key that unlocks the solutions to a vast number of these problems. It is not just another technique to learn; it is an essential tool for tackling advanced calculus and its many practical applications.

1.3 Prior Learning Connection

To effectively use the substitution method, a solid understanding of two prerequisite topics is essential.

1. **Differential Calculus (Chain Rule):** Integration by substitution is the direct inverse process of the chain rule used in differentiation. Understanding how the chain rule works to differentiate composite functions is crucial for recognizing the patterns needed to select the correct substitution when integrating.
2. **Standard Integration Formulas:** The entire purpose of the substitution method is to reduce a complicated integral to one of the standard forms (e.g., $\int x^n dx$, $\int \cos(x)dx$, $\int e^x dx$). These standard formulas must be memorized, as they represent the final step in solving the problem after a successful substitution has been made.

1.4 Core Definitions

The method of substitution is powerful enough to be used to derive the integration formulas for several key trigonometric functions. The following standard integrals are derived in the NCERT curriculum using this technique.

- **Standard Integral: $\int \tan(x)dx$**
  - **NCERT Reference:** Page 237
  - **Formula:** $\int \tan(x)dx = \log|\sec(x)| + C$
- **Standard Integral: $\int \cot(x)dx$**
  - **NCERT Reference:** Pages 237-238
  - **Formula:** $\int \cot(x)dx = \log|\sin(x)| + C$
- **Standard Integral: $\int \sec(x)dx$**
  - **NCERT Reference:** Page 238
  - **Formula:** $\int \sec(x)dx = \log|\sec(x) + \tan(x)| + C$
- **Standard Integral: $\int \text{cosec}(x)dx$**
  - **NCERT Reference:** Page 238
  - **Formula:** $\int \text{cosec}(x)dx = \log|\text{cosec}(x) - \cot(x)| + C$

We will now look more closely at how the NCERT textbook explains the principles and application of this method.

SECTION 2: WHAT NCERT SAYS

This section distills the core concepts, principles, and examples directly from the NCERT textbook for Class 12 Mathematics, Chapter 7. A thorough understanding of this material is crucial for academic success, as it forms the foundational basis of the board curriculum and examinations.

2.1 Key Statements

The method of integration by substitution is formally introduced in Section 7.3.1. The key principles can be summarized as follows:

1. **Transformation of Integrals:** An integral of the form $\int f(x)dx$ can be converted into a new form by changing the independent variable from x to t. This is achieved by defining x in terms of t through a substitution, such as $x = g(t)$.
2. **The Substitution Formula:** When the substitution $x = g(t)$ is made, where $dx = g'(t)dt$, the original integral transforms according to the formula: $\int f(x)dx = \int f(g(t))g'(t)dt$.
3. **The Primary Strategy:** The most common and effective strategy for applying this method is to inspect the integrand to find a composite function. The goal is to identify an inner function whose derivative also appears as a factor elsewhere in the integrand.
4. **Implicit Substitution:** In practice, the substitution is more frequently handled by setting a new variable t equal to a function of x, i.e., $t = g(x)$. Differentiating this expression yields $dt = g'(x)dx$, which allows for a direct replacement of the $g'(x)dx$ part of the integrand with dt.

2.2 Examples and Exercises

The NCERT textbook provides several worked examples that illustrate the application of these principles. Two representative examples are highlighted below.

- **Example 5 (ii), Page 236:**
  - **Problem:** $\int 2x \sin(x^2 + 1)dx$
  - **What it shows:** This example demonstrates the most direct type of substitution. Here, the inner function is $x^2 + 1$, and its derivative, 2x, is present as a distinct factor in the integrand, making it an ideal candidate for substitution.
  - **Why it's important:** It serves as a perfect illustration of the core pattern-recognition skill required for the substitution method and is the foundational problem type students should master first.
- **Example 6 (ii), Page 238:**
  - **Problem:** $\int(\sin(x) / \sin(x+a))dx$
  - **What it shows:** This example illustrates a more complex and strategic substitution. The substitution $t = x+a$ is chosen not because its derivative is present, but because it simplifies the structure of the denominator. Solving the problem then requires applying trigonometric identities to the numerator.

- **Why it's important:** It teaches that substitution is a versatile tool. It is not always limited to a simple function-derivative pair and can be used strategically to simplify the overall algebraic or trigonometric structure of the integrand.

- **Practice Problems:** NCERT Exercise 7.2, Questions 1-39 (Page 240-241).

Having reviewed the theory from the textbook, we can now synthesize this knowledge into a practical, step-by-step problem-solving framework.

SECTION 3: PROBLEM-SOLVING AND MEMORY

Moving from theory to application is key to mastering any mathematical concept. This section provides a structured method for solving problems using integration by substitution, highlights how the topic is approached in examinations, and shows how it connects to the broader mathematics curriculum.

3.1 Step-by-Step Method

The process demonstrated in the NCERT examples can be generalized into a reliable, step-by-step method for solving indefinite integrals by substitution.

- **Method: Integration by Substitution**
- **Pre-Check:** Analyze the integrand. Look for a composite function, f(g(x)), and check if the derivative of the inner function, g'(x), is also present as a factor. If this pattern is not immediately obvious, consider if a substitution could simplify the overall structure of the expression.
- **Core Steps:**
  1. **Choose Substitution:** Identify the inner function g(x) and set a new variable t = g(x).
  2. **Differentiate:** Find the derivative of the substitution with respect to x: dt/dx = g'(x). Rearrange this to solve for the differential term present in the integral, typically dt = g'(x)dx.
  3. **Substitute:** Replace every instance of g(x) in the integrand with t and the entire g'(x)dx term with dt. After this step, the integral must be expressed **completely in terms of t**. No x variables should remain; mixing variables is a critical error.
  4. **Integrate:** Solve the new, simpler integral with respect to t by applying the standard integration formulas.
  5. **Back-Substitute:** In the final result, replace t with the original function g(x) to express the answer in terms of the original variable x. Finally, add the constant of integration, C.

3.2 Exam Strategy

The scope of problems related to integration by substitution in the CBSE curriculum is well-defined by the NCERT textbook.

- **Worked Examples:** The fundamental techniques are demonstrated in Example 5 and Example 6 (Pages 236-239). These should be studied thoroughly.
- **Exercise Sets:** Mastery of the topic is achieved by practicing the problems in **Exercise 7.2 (Questions 1-39)**.
- **Problem Approach:** A recommended learning path is to start with problems where a function-and-derivative pair is obvious (e.g., Ex 7.2, Q1-Q4). After building confidence, progress to more complex problems that require algebraic or trigonometric manipulation *before* the substitution becomes apparent (e.g., Ex 7.2, Q32-Q35).

3.3 Topic Connections

Integration by substitution does not exist in isolation; it is deeply connected to other areas of the mathematics curriculum.

- **Prerequisites:** A strong command of **Differentiation (especially the Chain Rule)** is non-negotiable, as this skill is required to identify suitable substitutions. A thorough knowledge of the **Basic Indefinite Integrals** is also essential, as they are the building blocks needed to solve the simplified integral.
- **Forward Links:** This method is a foundational technique for more advanced integration methods. It is often used as an intermediate step within more complex procedures like **Integration by Partial Fractions** and **Integration by Parts**, making its mastery a prerequisite for success in the later sections of the unit.

3.4 Revision Summary

When revising this topic, focus on these critical points drawn directly from the NCERT framework.

1. The primary goal of the substitution method is to convert a complex integral into a standard, solvable form.
2. The method operates by changing the variable of integration from the original variable (e.g., x) to a new one (e.g., t).
3. The most effective strategy is to identify a function g(x) within the integrand whose derivative g'(x) is also present as a factor.
4. The standard substitution is t = g(x), which, after differentiation, implies that the term g'(x)dx can be replaced with dt.

5. After integrating with respect to the new variable t, you MUST substitute the original function of x back into the final expression.
6. For any indefinite integral, always remember to add the constant of integration, C, to represent the entire family of antiderivatives.
7. This method is the basis for deriving the standard integrals for the trigonometric functions tan(x), cot(x), sec(x), and cosec(x).