

Concept QuickStart – Integrals of Some Particular Functions

Unit 7: Integrals

Subject: For CBSE Class 12 Mathematics

SECTION 1: UNDERSTANDING THE CONCEPT

This section lays the groundwork for integrating a special class of functions. True mastery begins not with memorization, but with a deep understanding of the 'what' and 'why' behind these powerful calculus tools. We will explore the core concepts, their strategic importance, and their connection to your existing knowledge. Grasping this foundation is the essential first step toward applying these techniques with confidence and precision. This conceptual framework will prepare you for the specific methods detailed in your NCERT textbook.

1.1 What Is This Topic?

The central idea of this topic is to introduce a set of six fundamental integration formulas that serve as essential building blocks in calculus. These formulas provide direct solutions for integrands with specific quadratic structures, such as $1 / (x^2 - a^2)$, $1 / \sqrt{a^2 - x^2}$, and their variations.

However, these are not just arbitrary formulas. They are derived from foundational principles, including algebraic manipulation like **partial fractions** and **trigonometric substitution**. The primary goal is to learn how to recognize when a more complex integral can be transformed into one of these six standard forms.

Let's clear up a common misunderstanding right away: this topic is not solely about memorization. The true skill lies in the application—learning the techniques, such as **completing the square**, to reduce a wide range of integrals involving quadratic expressions into one of these solvable, standard forms.

1.2 Why It Matters

Mastering these particular integrals is strategically important because they provide a direct and efficient method for solving recurring patterns found throughout calculus. They are indispensable tools for the broader topic of integrating rational functions.

Their most powerful application lies in their ability to simplify seemingly difficult integrals. Any integral with a general quadratic expression in the denominator, of the form $\int dx / (ax^2 + bx + c)$, can be systematically converted into one of the standard forms and solved. This technique transforms a broad category of problems from complex challenges into straightforward

exercises. Mastering this technique is not just about this chapter; it's a foundational skill that will reappear when you tackle more advanced integration methods like partial fractions and in applications like finding the area under complex curves.

1.3 Prior Learning Connection

The derivation and application of these special integrals rely on several key skills you have already developed. A strong grasp of the following concepts is essential for success in this topic.

- **Algebraic Manipulation (Completing the Square):** This is the single most critical prerequisite. The method for solving integrals with general quadratic denominators ($ax^2 + bx + c$) depends entirely on your ability to algebraically transform the quadratic into the form of a sum or difference of two squares.
- **Trigonometric Substitutions:** The proofs for the standard formulas involving square roots (e.g., $\int dx / \sqrt{a^2 - x^2}$) are derived using trigonometric substitutions like $x = a \sin\theta$ or $x = a \tan\theta$. Understanding this connection provides deeper insight into where the inverse trigonometric and logarithmic results come from.
- **Integration by Substitution:** The core problem-solving process often involves an initial algebraic manipulation followed by a simple substitution (e.g., let $t = x + k$) to perfectly match the standard formula.

1.4 Core Formulas

The following six formulas are the foundation of this topic. For each problem you encounter, the initial goal is to manipulate the integrand to match one of these structures.

- **Formula (1)**
 - **NCERT Reference:** 7.4, Formula (1)
 - **Formula:** $\int dx / (x^2 - a^2) = (1/2a) \log |(x-a)/(x+a)| + C$
 - **Used In:** Problems where the denominator is a difference of two squares. This result arises from using **partial fractions** to decompose the integrand, leading to a logarithmic function.
- **Formula (2)**
 - **NCERT Reference:** 7.4, Formula (2)
 - **Formula:** $\int dx / (a^2 - x^2) = (1/2a) \log |(a+x)/(a-x)| + C$
 - **Used In:** Similar to Formula (1), this is the result of **partial fractions** applied to a denominator of the form $a^2 - x^2$.
- **Formula (3)**

- **NCERT Reference:** 7.4, Formula (3)
 - **Formula:** $\int dx / (x^2 + a^2) = (1/a) \tan^{-1}(x/a) + C$
 - **Used In:** Problems where the denominator is a sum of two squares. This formula yields an **inverse trigonometric function**, a result derived from **trigonometric substitution**.
- **Formula (4)**
 - **NCERT Reference:** 7.4, Formula (4)
 - **Formula:** $\int dx / \sqrt{(x^2 - a^2)} = \log |x + \sqrt{(x^2 - a^2)}| + C$
 - **Used In:** Problems with the reciprocal of a square root of $x^2 - a^2$. This produces a **logarithmic function** that includes the original square root term—a key pattern to recognize.
 - **Formula (5)**
 - **NCERT Reference:** 7.4, Formula (5)
 - **Formula:** $\int dx / \sqrt{(a^2 - x^2)} = \sin^{-1}(x/a) + C$
 - **Used In:** Problems with the reciprocal of a square root of $a^2 - x^2$. Like Formula (3), this yields an **inverse trigonometric function** and is proven using **trigonometric substitution**.
 - **Formula (6)**
 - **NCERT Reference:** 7.4, Formula (6)
 - **Formula:** $\int dx / \sqrt{(x^2 + a^2)} = \log |x + \sqrt{(x^2 + a^2)}| + C$
 - **Used In:** Problems with the reciprocal of a square root of $x^2 + a^2$. It produces a **logarithmic function** containing the original square root term, similar to Formula (4).

SECTION 2: WHAT NCERT SAYS

Here, we bridge theory and practice by examining the precise methods and worked examples from your NCERT textbook. Think of this as looking over the shoulder of an expert to see how the core formulas are put into action on problems that aren't in a simple, direct form. This section distills the textbook's problem-solving philosophy, showcasing how to systematically transform complex integrals into solvable ones. With the textbook's methods understood, we can now shift our focus to building a robust problem-solving toolkit.

2.1 Key Statements

The NCERT textbook outlines several key methods for solving problems that don't immediately match the six standard forms. The focus is on transforming the given integral into one that can be solved directly.

1. **Partial Fraction Decomposition:** The proof for the formula $\int dx / (x^2 - a^2)$ relies on decomposing the integrand $1 / (x^2 - a^2)$ into $(1/2a) [1/(x-a) - 1/(x+a)]$. This technique of breaking a rational function into simpler fractions is a fundamental strategy.
2. **Trigonometric Substitution:** For integrals involving square roots of quadratics, such as $\sqrt{(x^2 - a^2)}$, $\sqrt{(a^2 - x^2)}$, and $\sqrt{(x^2 + a^2)}$, the primary method used to derive the standard formulas is **trigonometric substitution** (e.g., letting $x = a \sec\theta$, $x = a \sin\theta$, or $x = a \tan\theta$).
3. **Completing the Square:** This is the most critical technique for applying the standard formulas. To evaluate an integral like $\int dx / (ax^2 + bx + c)$, the first step is to manipulate the general quadratic $ax^2 + bx + c$ into a sum or difference of two squares, such as $a[(x+k)^2 \pm m^2]$. This converts the problem into a recognizable, standard form.
4. **Handling Linear Numerators:** For integrals of the type $\int (px+q)dx / (ax^2+bx+c)$, the numerator $(px+q)$ is expressed as $A * (\text{derivative of denominator}) + B$. The purpose of this strategic split is to break the original integral into two manageable parts: one that can be solved instantly with a logarithmic substitution (since the numerator is the derivative of the denominator), and a second that is a constant over the quadratic, which is solved using the methods from this topic.

2.2 Examples and Exercises

The NCERT textbook uses worked examples to demonstrate the application of these methods.

- **Example 8**

- **Problem it solves:** Find (i) $\int dx / (x^2 - 16)$ and (ii) $\int dx / \sqrt{(2x - x^2)}$.
- **What it demonstrates:** Part (i) shows the most direct application of a standard formula ($\int dx / (x^2 - a^2)$, where $a=4$). Part (ii) demonstrates the need for algebraic manipulation (**completing the square** on $2x - x^2$ to get $1 - (x-1)^2$) before applying the standard formula for $\int dx / \sqrt{(a^2 - t^2)}$, where $a=1$ and $t = x-1$.
- **Why it's important:** It establishes the two primary scenarios: problems that are already in a standard form and problems that must first be converted into one.

- **Example 9**

- **Problem it solves:** Find (i) $\int dx / (x^2 - 6x + 13)$ and (ii) $\int dx / (3x^2 + 13x - 10)$.
- **What it demonstrates:** These examples provide a clear, step-by-step application of the **completing the square** method. Part (ii) specifically shows

the two-step process for handling quadratics where $a \neq 1$: first, factor out the leading coefficient (3) from the denominator, and *then* apply the completing the square method to the remaining expression.

- **Why it's important:** It provides a reliable template for solving any integral with a general quadratic expression in the denominator, which is a very common problem type.

- **Exercise 7.4**

- The specific exercises for this section are not available in the provided materials.

SECTION 3: PROBLEM-SOLVING AND MEMORY

Knowing the formulas is like knowing the names of the tools in a workshop. This section is where you become a craftsman. We will shift from theory to tactics, focusing on how to diagnose a problem, select the right method, and execute it with precision to avoid common mistakes. By focusing on pattern recognition and systematic procedures, you can develop both the speed and accuracy needed to excel.

3.1 Problem Types

Success with these integrals comes from quickly identifying the problem type and knowing the corresponding solution strategy. Here are the main categories derived from the NCERT text.

- **Problem Type: Direct Formula Application**

- **Structural Goal:** To directly apply one of the six standard formulas.
- **Recognition Cues:** The integral already matches one of the six forms: $\int dx / (x^2 \pm a^2)$, $\int dx / (a^2 - x^2)$, or their square root counterparts. The variable term is a simple x^2 .
- **What You're Really Doing:** Identifying the values of x and a and substituting them into the correct memorized formula.
- **NCERT References:** Section 7.4 Formulas (1)-(6), Example 8(i).
- **Confusable Types:** May be confused with general quadratics if you don't notice the absence of a bx term.

- **Problem Type: Integrals with General Quadratics**

- **Structural Goal:** To transform the integral $\int dx / (ax^2 + bx + c)$ or $\int dx / \sqrt{ax^2 + bx + c}$ into a standard, solvable form.

- **Recognition Cues:** The integral has a quadratic expression in the denominator that includes a linear (bx) term, preventing direct formula application.
 - **What You're Really Doing:** You are performing algebraic 'surgery' on the quadratic expression. '**Completing the square**' is the procedure you use to force the expression into a clean, standard form that matches one of the known formulas.
 - **NCERT References:** Method (7), Method (8), Example 8(ii), Example 9.
 - **Confusable Types:** Looks more complex than direct forms, but the underlying structure is the same once the square is completed.
- **Problem Type: Integrals with a Linear Numerator over a Quadratic Denominator**
 - **Structural Goal:** To solve integrals of the form $\int (px+q)dx / (ax^2+bx+c)$ and its square root variant.
 - **Recognition Cues:** The integral has a linear expression ($px+q$) in the numerator and a quadratic in the denominator.
 - **What You're Really Doing:** Splitting a complex problem into two manageable parts. The first part is designed to be a simple $\int (du/u)$ substitution, and the second part is engineered to become the 'General Quadratic' type we just mastered. It's a classic divide-and-conquer strategy.
 - **NCERT References:** Method (9).
 - **Confusable Types:** Can be confused with simpler substitution problems. Always check first if the numerator is a direct multiple of the denominator's derivative.

3.2 Step-by-Step Methods

Here is a detailed breakdown of the most common procedure you will use: integrating a function with a general quadratic in the denominator.

- **Type: Integrating General Quadratics [$\int dx / (ax^2 + bx + c)$]**
 - **Pre-Check:** Is the denominator a quadratic with a linear (bx) term? Can it be factored easily? If not, this method is likely necessary.
 - **Core Steps:**
 1. **Make the coefficient of x^2 unity.** Factor out the coefficient 'a' from the denominator: $1/a \int dx / (x^2 + (b/a)x + c/a)$.
 2. **Complete the square.** Focus on the terms $x^2 + (b/a)x$. (Take half the coefficient of x , square it, and then add and subtract it *inside the*

denominator to keep the expression balanced.) This gives $x^2 + (b/a)x + (b/2a)^2 - (b/2a)^2 + c/a$.

3. **Simplify the expression.** The first three terms form a perfect square $(x + b/2a)^2$. Combine the constant terms. The denominator is now in the form $(x+k)^2 \pm m^2$.
 4. **Substitute and Solve.** Let $t = x + b/2a$, so $dt = dx$. The integral is now transformed into a standard form like $1/a \int dt / (t^2 \pm m^2)$, which you can solve using the appropriate formula from Section 1.4. Finally, substitute the original expression for t back into your result to give the answer in terms of x .
- **Variants:** If the quadratic is under a square root ($\int dx / \sqrt{ax^2 + bx + c}$), the exact same steps for completing the square apply. The final integral will simply match one of the standard square root forms.
 - **When NOT to Use:** This method is unnecessary if the numerator is the derivative (or a multiple of it) of the denominator. In that case, a simple substitution $t = ax^2 + bx + c$ is much faster.

3.3 How to Write Answers

Presenting your solution clearly and logically is crucial for earning full marks. Follow this template for problems that require manipulation before applying a formula.

- **Answer Template:** A structured flow from the original problem to the final, simplified answer.
- **When to Use:** This template is ideal for any problem requiring algebraic steps like completing the square or expressing the numerator in a new form.
- **Line-by-Line:**
 - **L1 (Setup):** Write the original integral clearly. Let this be $I = \int \dots$
 - **L2 (Manipulation):** Show the key algebraic steps. If completing the square, show the addition and subtraction of the term and the resulting perfect square form.
 - **L3 (Substitution):** If a substitution is needed, state it explicitly. For example: "Let $t = x - 3$, then $dt = dx$."
 - **L4 (Application):** State the standard formula you are using. For example: "Using the formula $\int dx/(x^2+a^2) = (1/a)\tan^{-1}(x/a) + C$, we get:"
 - **L5 (Solution):** Write the final answer in terms of the original variable (x) and do not forget to add the constant of integration, $+ C$.

- **Essential Phrases:**
 - "Completing the square for the denominator, we get:"
 - "Let $t = \dots$, so $dt = \dots$ "
 - "Using the formula $\int \dots =$ "
- **General Rules:** In an exam setting, these are the non-negotiables that separate a good answer from a perfect one.
 - • **Always add '+ C'** for indefinite integrals.
 - • **Clearly define** any variables used for substitution (t , u , etc.).
 - • **Show intermediate algebraic steps.** Do not jump from the problem to the final standard form without justification.
 - • Use absolute value signs inside logarithms (e.g., $\log|\dots|$) as required by the formulas.

3.4 Common Mistakes

Being aware of common pitfalls is the fastest way to avoid them. Here are the three most frequent errors I see students make:

- **Pitfall #1: Coefficient Error in Formula Application**
 - **Category:** Logic / Formatting
 - **Occurs In:** Applying the final standard formula.
 - **I. Wrong:** Forgetting the leading $1/2a$ or $1/a$ term in the formulas for $\int dx/(x^2-a^2)$, $\int dx/(a^2-x^2)$, or $\int dx/(x^2+a^2)$.
 - **✓ Fix:** Memorize the formulas completely, including their coefficients. Write the formula down on your paper before substituting values into it as a final check.
- **Pitfall #2: Algebraic Error in Completing the Square**
 - **Category:** Algebra
 - **Occurs In:** The initial manipulation of the quadratic denominator.
 - **I. Wrong:** Making errors when factoring out the leading coefficient 'a' or when calculating the constant term to add and subtract.
 - **✓ Fix:** Perform the algebraic steps carefully and deliberately. After completing the square, mentally expand your result to ensure it matches the original quadratic.
- **Pitfall #3: Formula Mix-up**

- **Category:** Logic
- **Occurs In:** Choosing which of the six standard formulas to apply.
- **Wrong:** Using the \sin^{-1} formula for a $\sqrt{x^2 - a^2}$ type, or mixing up the log forms for $x^2 - a^2$ and $a^2 - x^2$.
- **Fix:** Pay close attention to the signs and the variable order ($x^2 - a^2$ vs. $a^2 - x^2$). Create flashcards for the six formulas and practice until recognition is instant.

3.5 Exam Strategy

A systematic approach to learning and revision will build confidence for the exam.

- **Example Range:** Master the solved examples in NCERT Section 7.4, specifically **Examples 8 and 9**. They cover the core applications of direct use and completing the square.
- **Exercise Sets:** The exercises for Section 7.4 are not included in the source material.
- **Question Patterns:** Direct questions like $\int dx / (x^2 - 9)$ are rare. Expect questions that require one or two steps of manipulation first. The most common pattern involves a general quadratic ($ax^2 + bx + c$) that you must first transform by completing the square.
- **Approach:** Follow this learning progression:
 1. **Memorize the Six Formulas:** Use flashcards or repetition until they are second nature.
 2. **Practice Completing the Square:** Work on this algebraic skill in isolation until you are fast and accurate.
 3. **Combine and Apply:** Solve problems of the form $\int dx / (ax^2 + bx + c)$ and its variants, focusing on the full process from manipulation to final answer.

3.6 Topic Connections

Understanding how this topic fits into the larger curriculum helps reinforce its importance.

- **Prerequisites:**
 - **Basic Integration Formulas:** All standard integrals (for x^n , $\sin x$, e^x , etc.) are the building blocks.
 - **Trigonometric Identities:** These are essential for understanding the derivations of several formulas.
 - **Algebra of Quadratic Expressions:** Proficiency in factoring and completing the square is non-negotiable.
- **Forward Links:**

- • **Integration by Partial Fractions:** When a rational function has an irreducible quadratic factor in the denominator, the process of breaking it down will result in an integral of the form $\int(Ax+B)dx/(ax^2+bx+c)$, which requires the methods learned here to solve.
- • **Definite Integrals:** These formulas are frequently used as the final step in evaluating definite integrals, which are used to calculate area under curves, volumes, and in various physics and engineering applications.

3.7 Revision Summary

Use this concise summary for quick revision and to reinforce the most critical concepts.

- **Key Points:**

1. This topic provides six special formulas for integrating functions of the form $1/(x^2 \pm a^2)$, $1/(a^2 - x^2)$, and their counterparts with square roots.
2. Mastering the technique of **completing the square** is the most critical skill, as it transforms general quadratics (ax^2+bx+c) into one of the standard forms.
3. For forms with a linear term over a quadratic, $\int(px+q)dx / (ax^2+bx+c)$, the strategy is to express the numerator as **A × (derivative of denominator) + B**.
4. **Trigonometric substitution** is the underlying method used to derive the formulas involving square roots.
5. Pay close attention to the leading coefficients in the final formulas, such as $1/2a$ and $1/a$, as they are common sources of error.
6. Always add the **constant of integration, + C**, to the final answer for any indefinite integral.

- **Memory Aids: Expert Tip: Match the Factors to the Log** To remember the logarithmic formulas for $\int dx/(x^2 - a^2)$ and $\int dx/(a^2 - x^2)$:

- For the denominator $x^2 - a^2$, the factors are $(x-a)$ and $(x+a)$. The log contains $|(x-a)/(x+a)|$.
- For the denominator $a^2 - x^2$, the factors are $(a-x)$ and $(a+x)$. The log contains $|(a+x)/(a-x)|$.

- The argument of the logarithm is simply the ratio of the factors of the denominator.

Your journey through this topic is a perfect illustration of how mathematical mastery is built. You began with six fundamental formulas—the bedrock. From there, you learned the art of algebraic transformation, turning complex quadratics into these familiar forms. Finally, you

saw how even more intimidating integrals can be strategically broken down into these solvable parts. These skills are more than just a chapter in a textbook; they are a gateway to a deeper understanding of calculus, equipping you to solve a vast new range of problems in mathematics and beyond. Practice these techniques, and you will find them becoming an indispensable part of your analytical toolkit.



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