CONCEPT QUICKSTART – Differentiability

Unit: Unit 5: Continuity and Differentiability

Subject: CBSE Class 12 Mathematics

SECTION 1: UNDERSTANDING THE CONCEPT

In Class 11, our mathematical journey focused largely on the differentiation of simple polynomial and basic trigonometric functions. As we transition into Class 12, the curriculum shifts toward a more sophisticated analysis of transcendental functions—such as inverse trigonometric, exponential, and logarithmic functions—and composite functions. Differentiability is presented as the "higher-order" cousin of continuity; while continuity ensures a function has no breaks or jumps, differentiability ensures the function is "smooth" enough to have a defined slope or rate of change at any given point.

1.1 What Is Differentiability? Differentiability refers to the existence of a unique derivative for a function at a specific point in its domain. The "Big Idea" is that for a function to be differentiable at a point c, the rate of change must be consistent and finite whether you approach c from the left or the right.

From a professional perspective, the derivative is fundamentally a limit representing the instantaneous rate of change. A common misunderstanding among students is the belief that if a function is defined at a point, it must be differentiable there. This is incorrect. A classic counter-example is the modulus function $f(x) = |x|$. While it is continuous at $x = 0$, it is not differentiable there because the graph has a "sharp corner." Visually, a "smooth curve" allows for a single tangent, but a "sharp corner" creates an ambiguity where the limit from the left (-1) does not equal the limit from the right (1), causing the limit to fail.

1.2 Why It Matters Differentiability is the vital link that connects algebraic equations to geometric reality, allowing us to calculate the exact slope of a tangent to a curve. Beyond geometry, mastering this concept serves as the essential gateway to "exponential and logarithmic techniques" introduced in Section 5.1 of the NCERT text. These functions provide powerful differentiation techniques that simplify complex products and quotients that would otherwise be nearly impossible to solve.

1.3 Prior Learning Connection To master differentiability, students must have a firm grasp of the following logical necessities:

- **Limits:** The formal definition of a derivative is itself a limiting process; without limits, the "instantaneous" nature of change cannot be defined.

- **Continuity:** A logical prerequisite; as established in Theorem 3, a function must be connected (continuous) before it can be smooth (differentiable).
- **Polynomial Differentiation:** Basic rules like $(x^n)' = nx^{n-1}$ serve as the building blocks for the more complex nested operations in Class 12.

1.4 Core Definitions The following definitions and theorems form the structural backbone of the NCERT curriculum:

- **[Derivative of a function at a point c]**
  - NCERT Reference: Section 5.3
  - Definition: $f'(c) = \lim_{h \to 0} (f(c + h) - f(c)) / h$
  - Used In: First principle calculations and proving the non-existence of a derivative.
- **[Differentiability in an Interval]**
  - NCERT Reference: Section 5.3
  - Definition: A function is differentiable in an interval [a, b] if it is differentiable at every point in (a, b), with the right-hand derivative at 'a' and left-hand derivative at 'b' being finite and equal.
  - Used In: Establishing the domain for Mean Value Theorems and general calculus applications.
- **[Theorem 3: Differentiability implies Continuity]**
  - NCERT Reference: Section 5.3
  - Definition: If a function f is differentiable at a point c, then it is also continuous at c. The proof relies on the identity: $f(x) - f(c) = [(f(x) - f(c)) / (x - c)] \cdot (x - c)$.
  - Used In: Proving the relationship between the "smoothness" and "connectedness" of functions.
- **[Theorem 4: Chain Rule]**
  - NCERT Reference: Section 5.3.1
  - Definition: If $f = v \circ u$, then $dy/dx = (dv/dt) \cdot (dt/dx)$, where $t = u(x)$.
  - Used In: Differentiating nested or "function of a function" structures.

These theoretical definitions are not mere formalities; they are the structural supports upon which all practical problem-solving in the NCERT syllabus is built.

SECTION 2: WHAT NCERT SAYS

The NCERT textbook prioritizes the hierarchical relationship between continuity and differentiability. It establishes that while differentiability is a stricter condition, it provides the necessary "smoothness" required for the advanced mechanics of the Chain Rule.

2.1 Key Statements

- **The Differentiable $\Longrightarrow$ Continuous Rule:** Every differentiable function is continuous (Corollary 1), but the converse is not true (the modulus function is the standard counter-proof).
- **Chain Rule Mechanics:** Differentiation of composite functions requires breaking the function into layers; the derivative is the product of the derivative of the outer layer w.r.t. the inner function and the derivative of the inner function w.r.t. x.
- **Implicit Function Rule:** In relationships where y cannot be easily isolated (e.g., $x + \sin(xy) - y = 0$), we differentiate every term w.r.t. x and solve for dy/dx.
- **Algebra of Derivatives:** The sum, difference, product (Leibniz rule), and quotient rules remain the standard operational framework for all Class 12 problems.

2.2 Examples and Exercises High-value targets for exam preparation include:

- **Example 21 (Page 121):** Derivative of $\sin(x^2)$. *Conceptual Goal:* Demonstrating the basic "layer" approach of the Chain Rule.
- **Example 23 (Page 123):** Differentiating $y + \sin(y) = \cos(x)$. *Conceptual Goal:* Mastering Implicit Differentiation where y appears on both sides.
- **Example 24 (Page 124):** Derivative of $\sin^{-1} x$. *Conceptual Goal:* High-value exam target. It emphasizes the domain restriction x \in (-1, 1) and uses the identity $\cos^2 y = 1 - \sin^2 y$ to reach the final form $1/\sqrt{(1 - x^2)}$.

Exercise Ranges:

- **Exercise 5.2 (Q1-Q8):** Foundational. Focuses on the Chain Rule and standard functions.
- **Exercise 5.2 (Q9-Q10):** Differentiability Proofs. These are critical targets involving the **Modulus function** and the **Greatest Integer Function**, specifically testing non-differentiability at "sharp" or "broken" points.
- **Exercise 5.3 (Q1-Q15):** Composite & Inverse. Covers implicit differentiation and the standard forms of inverse trigonometric functions.

Understanding "what" the book says allows us to now focus on "how" to efficiently solve these problems in an exam setting.

SECTION 3: PROBLEM-SOLVING AND MEMORY

Mastering differentiability is not about rote memorization, but about identifying function structures—Implicit, Composite, or Inverse—to apply the correct strategic approach.

3.1 Problem Types

- **Problem Type: Composite Functions (Chain Rule)**
    - Structural Goal: Finding the derivative of a "nested" function.
    - Recognition Cues: Surface [f(x) is inside another function] | Structural [e.g., tan(√x) or sin(cos(x))].
    - What You're Really Doing: Differentiating from the outside-in, layer by layer.
    - NCERT References: Example [21] | Exercise [5.2, Q1-8].
    - Confusable Types: Product rule problems where two distinct functions are multiplied (x^2 \cdot sin(x)) rather than nested.
- **Problem Type: Implicit Differentiation**
    - Structural Goal: Finding dy/dx when y cannot be isolated.
    - Recognition Cues: Surface [y is inside a trigonometric function] | Structural [x and y are mixed, e.g., x + sin(xy) - y = 0].
    - What You're Really Doing: Applying the Chain Rule to y as a function of x and then algebraically grouping dy/dx terms.
    - NCERT References: Examples [22, 23] | Exercise [5.3, Q1-8].
    - Confusable Types: Explicit functions like $y = x^2 + 5$, where y is already isolated.
- **Problem Type: Inverse Trigonometric Differentiation**
    - Structural Goal: Finding derivatives of $\sin^{-1} x$, $\cos^{-1} x$, etc.
    - Recognition Cues: The presence of the $^{-1}$ notation.
    - What You're Really Doing: Letting $y = f^{-1}(x)$, rewriting as f(y) = x, and using implicit differentiation.
    - NCERT References: Example [24] | Exercise [5.3, Q9-15].

3.2 Step-by-Step Methods

- **Type: Chain Rule Solution Method**
    - Pre-Check: Identify the "inner" function t = u(x) and the "outer" function v(t).

- Core Steps:
  1. **Identify components:** In $\sin(x^2)$, outer is $\sin(t)$, inner is x^2.
  2. **Differentiate outer:** Find dv/dt (e.g., $\cos(t)$).
  3. **Differentiate inner:** Find dt/dx (e.g., $2x$).
  4. **Multiply and Substitute:** Multiply results and replace t with the original x-expression ($\cos(x^2) \cdot 2x$).
- Variants: For three functions ($w \circ u \circ v$), simply add a third multiplication step for the innermost layer.
- When NOT to Use: Simple polynomial terms like $x^3 + 5x$ where functions are added, not nested.

3.3 How to Write Answers

Answer Template: The Limit Comparison Frame (For Differentiability Proofs)

- **Line 1:** State the function f(x) and the point c.
- **Line 2:** Calculate **Left Hand Derivative (LHD)**: $\lim_{h\to 0^{-}} (f(c + h) - f(c)) / h$.
- **Line 3:** Calculate **Right Hand Derivative (RHD)**: $\lim_{h\to 0^{+}} (f(c + h) - f(c)) / h$.
- **Line 4:** State Conclusion: "Since LHD ≠ RHD at x = c, the function is not differentiable" OR "Since LHD and RHD are finite and equal, f is differentiable."

Essential Phrases:

- "Differentiating both sides with respect to x..."
- "By applying the Chain Rule..."
- "Since the function is discontinuous at x = c, it is not differentiable by Theorem 3."

General Presentation Rules:

1. Always explicitly mention "Differentiating w.r.t. x."
2. Use dy/dx and f'(x) interchangeably but stay consistent within one problem.
3. Group all dy/dx terms on the left side before factoring.

3.4 Common Mistakes

- **Pitfall [1]: Forgetting the Inner Derivative**
  - Category: Logic. Occurs in Chain Rule Step 3.
  - Wrong: Differentiating $\cos(3x)$ as $-\sin(3x)$.

    - ✓ Fix: Always write d/dx of the "inner argument" immediately after the outer derivative (Result: -sin(3x) \cdot 3).
- **Pitfall [2]: Assuming Differentiability from Definition**
    - Category: Logic.
    - Wrong: Thinking f(x) is differentiable just because f(c) exists.
    - ✓ Fix: Always check for "sharp corners" or discontinuities.
- **Condition [1]: Continuity Requirement**
    - Rule: A function must be continuous to be differentiable. If you prove a function is discontinuous (e.g., a Jump Discontinuity in a piecewise function), you can immediately conclude it is non-differentiable.

3.5 Exam Strategy Master the "Strategic Transition": Start by perfecting the mechanics of the Chain Rule in Exercise 5.2. Once your differentiation is fluid, move to the mixed Implicit and Inverse problems in Exercise 5.3. This prevents "algebraic fatigue" from interfering with your conceptual understanding.

3.6 Topic Connections

- **Backward Link:** Relies on "Limits of Trigonometric Functions" and Class 11 continuity rules.
- **Forward Link:** Gateway to "Exponential and Logarithmic Differentiation." These functions offer "powerful techniques" to handle variables in exponents and simplify massive products into manageable sums.

3.7 Revision Summary

- Standard Derivative: $(x^n)' = nx^{n-1}$
- Standard Derivative: (sin x)' = cos x
- Standard Derivative: (cos x)' = -sin x
- Standard Derivative: $(\tan x)' = \sec^2 x$
- Inverse Trig: $(\sin^{-1} x)' = 1 / \sqrt{(1 - x^2)}$
- Inverse Trig: $(\tan^{-1} x)' = 1 / (1 + x^2)$
- Fundamental Rule: Differentiable $\implies$ Continuous.

Checklist for Chain Rule:

1. Identify all nested layers (Work outside-in).
2. Take the derivative of the outer layer.

3. Multiply by the derivative of the inner layer.
4. Simplify the final expression back into terms of x.

By mastering differentiability, you transform static functions into dynamic rates of change, turning the "sharp corners" of algebra into the "smooth curves" of calculus.