CONCEPT QUICKSTART – Continuity

Unit: Unit 5: Continuity and Differentiability

Subject: For CBSE Class 12 Mathematics

SECTION 1: UNDERSTANDING THE CONCEPT

Continuity serves as the indispensable analytical bridge between the foundational limits explored in Class 11 and the sophisticated mechanics of differential calculus in Class 12. It transforms the study of functions from a collection of isolated points into a cohesive examination of smooth transitions. Mastery of this concept is non-negotiable for Class 12 success because the logical framework of the entire Calculus syllabus—including the validity of derivatives and integrals—presupposes that a function is continuous. In the CBSE Board Exam, a failure to rigorously prove continuity often results in a total loss of marks for subsequent steps in a problem.

1.1 What Is Continuity? The "Big Idea" behind continuity is predictability. A function is continuous if we can predict its value at a specific point by observing its behavior in the immediate neighborhood of that point. Mathematically, this means the values of the function converge exactly to the value assigned at that point. A common misunderstanding is that "lifting the pen while drawing" is the only definition of discontinuity. This is naively incomplete. As a Senior Educator, I must emphasize that a function can only be discontinuous at a point within its domain. For example, $f(x) = 1/x$ is technically a continuous function because it is continuous at every point where it is defined; $x = 0$ is simply not in its domain, so we do not label it a "point of discontinuity" in the rigorous sense.

1.2 Why It Matters Continuity is the cornerstone of mathematical analysis because it ensures that a function's behavior is stable. If a function is continuous, its graph has no sudden jumps or holes at points where the function exists. This stability is the primary gatekeeper for the rest of the unit; according to Theorem 3, differentiability requires continuity. If you cannot prove a function is continuous, you cannot mathematically justify its derivative.

1.3 Prior Learning Connection To master Continuity, a student must be fluent in the following Class 11 concepts:

- **Limits (LHL and RHL):** Since the formal definition of continuity requires the limit to exist, students must be able to calculate Left Hand Limits and Right Hand Limits. So What? Because if LHL ≠ RHL, the limit doesn't exist, and the function is immediately discontinuous.

- **Domain of Functions:** Continuity is evaluated at points in the domain. So What? Knowing the domain prevents you from incorrectly identifying points where the function is undefined as "points of discontinuity."
- **Algebra of Functions:** You must know how to combine functions. So What? Board questions often combine a polynomial and a trigonometric function (e.g., f(x) = x + sin x). You must justify continuity by citing that the sum of two continuous functions is also continuous.

1.4 Core Definitions

• **Definition of Continuity at a Point** • NCERT Reference: Section 5.2, Definition 1 •

Definition: $\lim_{x \to c} f(x) = f(c)$ • Used In: Checking Continuity at a Point; Basic Verification

Problems.

• **Elaborate Point Continuity** • NCERT Reference: Page 105 • Definition: A function is continuous at x = c if the left hand limit, right hand limit, and the value of the function f(c) exist

and are equal: $\lim_{x \to c-} f(x) = \lim_{x \to c+} f(x) = f(c)$ • Used In: Piecewise Function Analysis;

Modulus Function Problems.

• **Continuity of a Function & Endpoints** • NCERT Reference: Section 5.2, Definition 2 (Page 107) • Definition: A real function f is continuous if it is continuous at every point in its domain. For an interval [a, b], f is continuous if it is continuous at every point in (a, b) and: $\lim_{x \to a+} f(x) = f(a)$ AND $\lim_{x \to \beta-} f(x) = f(b)$. • Used In: Discussing continuity over closed intervals.

• **Algebra of Continuous Functions** • NCERT Reference: Page 113–114, Theorem 1 & Remarks • Definition: If f and g are continuous at c, then (f + g), (f – g), (f . g) are continuous at c. Also, (f/g) is continuous if g(c) ≠ 0. Remarks: (λ . g) is continuous for any real λ, and (1/g) is continuous wherever g(x) ≠ 0. • Used In: Proving continuity of rational and combined functions.

• **Continuity of Composite Functions** • NCERT Reference: Page 115, Theorem 2 • Definition: If g is continuous at c and f is continuous at g(c), then (f ∘ g) is continuous at c. • Used In: Proving continuity for functions like $\sin(x^2)$ or |1 – x + |x||.

The preceding theoretical foundations establish the rigorous criteria used by the NCERT curriculum to define mathematical "smoothness."

SECTION 2: WHAT NCERT SAYS

Aligning strictly with the NCERT framework is essential for Board Exam success, as the terminology, proof structures, and specific example types provided in the textbook form the basis for the official marking schemes.

2.1 Key Statements

1. **Identity and Constant Functions:** The identity function $f(x) = x$ and any constant function $f(x) = k$ are continuous at every real number.
2. **Polynomial Continuity:** Every polynomial function is continuous across its entire domain ($\mathbb{R}$).
3. **Rational Function Continuity:** A rational function $f(x) = p(x)/q(x)$ is continuous at every point where the denominator $q(x) \neq 0$.
4. **Trigonometric Continuity:** Sine and Cosine functions are continuous everywhere. Tangent is continuous for all $x \neq (2n + 1)\pi/2$.
5. **Composition Rule:** $(f \circ g)(x) = f(g(x))$ is continuous if g is continuous at c and f is continuous at g(c).

2.2 Examples and Exercises

- **Example 3 (Page 106):** Continuity of $f(x) = |x|$ at $x = 0$. This is strategically important because it demonstrates the use of LHL and RHL at the "turning point" of the modulus.
- **Example 15 (Page 112):** Analyzes the Greatest Integer Function $f(x) = [x]$. This proves discontinuity at every integral point—a high-yield concept for MCQ and 2-mark questions.
- **Example 17 (Page 114):** Proof of sine function continuity. Strategically critical because it introduces the "Substitution Method": putting $x = c + h$ such that as $x \to c$, $h \to 0$.

Exercise 5.1 Categorization:

- **Q1–Q13 (Foundational):** Direct verification at specified points.
- **Q14–Q16 (Procedural):** Piecewise functions with multiple intervals.
- **Q17–Q30 (High-Yield):** Finding unknown constants (k, a, or b). These are the most recurring patterns in CBSE Board Exams.

Transitioning from "what to know" to "how to solve" requires a methodical approach to different problem structures.

SECTION 3: PROBLEM-SOLVING AND MEMORY

Mastering Continuity requires moving beyond definitions into procedural recognition and error prevention.

3.1 Problem Types

• **Problem Type: Verification at a Specified Point** • Structural Goal: Prove $\lim_{x \to c} f(x) = f(c)$. • Recognition Cues: "Check the continuity of f at x = [number]." • What You're Really Doing: Checking if f(c) is defined and if the limit exists and matches that value. • NCERT References: Examples 1, 2, 5, 6; Exercise 5.1, Q1, Q2. • Confusable Types: Identifying "all points of discontinuity" (requires checking the entire domain, not just one point).

• **Problem Type: Finding Points of Discontinuity** • Structural Goal: Find all x where f is not continuous. • Recognition Cues: "Find all points of discontinuity..." • What You're Really Doing: Checking "junction points" in piecewise functions or integral points for [x]. • NCERT References: Examples 11, 15; Exercise 5.1, Q6–Q12. • Confusable Types: Points outside the domain. Students often incorrectly list x = 0 as a point of discontinuity for f(x) = 1/x, but since $0 \notin$ Domain, it is not a point of discontinuity.

• **Problem Type: Finding Unknown Constants (The "Exam Favorite")** • Structural Goal: Solve for variables like k, a, or b. • Recognition Cues: "Find the value of k such that f is continuous at x = c." • What You're Really Doing: Forcing the condition LHL = RHL = f(c) to create an algebraic equation. • NCERT References: Exercise 5.1, Q26–Q30.

3.2 Step-by-Step Methods

• **Type: Checking Point Continuity: Solution Method** • Pre-Check: Is the point c in the domain? If not, continuity cannot be discussed. • Core Steps: Step 1: Calculate the value of the function f(c). Step 2: Calculate $\lim_{x \to c} f(x)$. Use LHL and RHL if the function is piecewise. Step 3: If LHL = RHL = f(c), conclude f is continuous at x = c. • Variants: Using substitution x = c + h for trigonometric proofs.

• **Type: Finding Unknown Constants: Solution Method** • Pre-Check: Identify the junction point c where the function's definition changes. • Core Steps: Step 1: Write the condition for continuity: "Since f is continuous at x = c, LHL = RHL = f(c)." Step 2: Calculate LHL using $\lim_{x \to c^-} f(x)$ and RHL using $\lim_{x \to c^+} f(x)$. Step 3: Equate the results from Step 2 to f(c) and solve for the unknown (k, a, or b). • When NOT to Use: Do not use if the function is already a simple polynomial without a junction point.

3.3 How to Write Answers

• **Answer Template: Standard Continuity Proof** • When to Use: All board exam questions asking to "examine" or "find k." • Line-by-Line: L1: "The function f is defined at x = c." L2: "f(c) = [value]." L3: "LHL = $\lim_{x \to c^-} f(x)$ = [calculation] and RHL = $\lim_{x \to c^+} f(x)$ = [calculation]." L4: "Since LHL = RHL = f(c), the function is continuous at x = c." • Essential Phrase: "Since f is a polynomial function, it is continuous for all $x \in \mathbb{R}$." (Use this for full marks when justifying general continuity).

• **General Rules for Formatting:**

1. Always state the domain explicitly if it is restricted.
2. Use the limit notation $\lim_{x \to c}$ in every line of your limit calculation.
3. The board awards 1 mark for the explicit statement LHL = RHL = f(c).

3.4 Common Mistakes

• **Pitfall 1: Forgetting the Value of the Function** • Category: Logic. • Wrong: Only proving that the limit exists (LHL = RHL) and assuming that is enough. • ✓ Fix: You must calculate f(c). If LHL = RHL but they do not equal f(c), the function is discontinuous.

• **Pitfall 2: Incorrect Interval Analysis** • Category: Logical Rigor. • Wrong: Only checking junction points when asked to "Discuss continuity of f." • ✓ Fix: You must define Case 1 ($x < c$), Case 2 ($x > c$), and Case 3 ($x = c$).

Critical Conditions:

1. Endpoints: Continuity at 'a' in [a, b] only requires the Right Hand Limit: $\lim_{x \to a^+} f(x) = f(a)$.
2. Domain: A function is "continuous" if it is continuous at every point in its domain.

3.5 Exam Strategy

- **High-Yield Practice:** Focus on Exercise 5.1, Q17, Q18, and Q26–Q30. These "Find k/a/b" questions are the most common 4-mark questions.
- **Question Patterns:** Recurring patterns include checking continuity of |x| and [x].
- **Approach:** Master polynomial checks → Master piecewise junction checks → Master unknown constant finding.

3.6 Topic Connections

- **Prerequisites:** Limits (Class 11) for calculating LHL/RHL; Algebra of Functions for justifying sum/product continuity.
- **Forward Links:** Differentiability (Theorem 3: Every differentiable function is continuous); Integration (Continuity is usually required for a function to be integrable).

3.7 Revision Summary

The 3-Point Success Checklist:

1. Is the function defined at x = c? (f(c) must exist).
2. Does the limit exist? (LHL must equal RHL).
3. Do they match? (Limit must equal f(c)).

Key Facts:

1. $\lim_{x \to c} f(x) = f(c)$ is the ultimate definition.
2. Polynomial, Identity, and Constant functions are continuous for all $\mathbb{R}$.
3. For a closed interval [a, b], we only check the one-sided limit at the endpoints.
4. The Greatest Integer Function [x] is discontinuous at every integer.
5. The sum, difference, and product of two continuous functions are continuous.
6. The quotient f/g is continuous provided $g(x) \neq 0$.
7. The composition $(f \circ g)$ is continuous if g is continuous at c and f is continuous at g(c).
8. Every differentiable function is continuous (Theorem 3).
9. The converse is NOT true: |x| is continuous at 0 but not differentiable at 0.
10. A "continuous function" is one that is continuous at every point in its domain.

Mastering the logic of Continuity provides the structural integrity needed to explore the dynamic world of Differentiability and the broader landscape of Calculus.