CONCEPT QUICKSTART – Shortest Distance between Two Lines

Unit: Unit 11: Three Dimensional Geometry

Subject: CBSE Class 12 Mathematics Students

SECTION 1: UNDERSTANDING THE CONCEPT

In our previous studies of 2D geometry, lines were limited to two behaviors: they either crossed at a point or remained parallel. As we move into 3D space, the moving power of mathematical invention shifts from reasoning to "imagination," as noted by A. De Morgan. In the 3D world, lines can pass each other without ever meeting and without being parallel—these are called "skew lines." Understanding the spatial separation between such lines is a cornerstone of modern engineering, from the work of Leonhard Euler to modern flight path management. While 2D distance is a simple perpendicular drop, finding the shortest distance in 3D requires a sophisticated vector-based approach to identify the absolute minimum gap between two objects moving in different planes.

1.1 What Is Shortest Distance between Two Lines? The "Big Idea" is that the shortest distance between any two lines in space is the length of the unique line segment that is mutually perpendicular to both lines. Professionally, this represents the minimum possible displacement between any point on the first line and any point on the second. Students often mistakenly assume that non-parallel lines must eventually intersect; however, in 3D, most lines are "skew," meaning they are non-coplanar and will never meet regardless of how far they are extended.

1.2 Why It Matters This concept bridges the gap between abstract vector algebra and physical visualization. By calculating the "Shortest Distance," we apply vector cross products to solve tangible problems, such as ensuring safe clearance between aircraft flight paths or determining the minimum length of a structural support connecting two non-intersecting bridge cables. It is the pinnacle of CBSE Class 12 spatial reasoning.

1.3 Prior Learning Connection To master this topic, you must recall these foundational tools from Chapter 10:

- **Vector Cross Products ($b_1 \times b_2$):** Essential because the line of shortest distance is perpendicular to both given lines. The cross product is the mathematical tool used to find this "common perpendicular."
- **Position Vectors (a_1 and a_2):** These identify specific "anchor points" on each line, allowing us to create a displacement vector (a_2 - a_1) between them.
- **Dot Products:** Used to find the projection of the displacement vector onto the common perpendicular, which yields the actual distance value.

1.4 Core Definitions The following formal items from NCERT Section 11.2 and 11.5 are the building blocks for all exam problems:

[Normalization of Direction Ratios] NCERT Reference: Section 11.2 Formula: $l = \pm a/\sqrt{(a^2 + b^2 + c^2)}$, $m = \pm b/\sqrt{(a^2 + b^2 + c^2)}$, $n = \pm c/\sqrt{(a^2 + b^2 + c^2)}$ Used In: Converting direction ratios to unique direction cosines.

[Distance between Two Points] NCERT Reference: Section 11.2.1 Formula: $d = \sqrt{((x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2)}$ Used In: Finding the magnitude of vectors and basic coordinate distance.

[Skew Lines] NCERT Reference: Section 11.5 Definition: Lines in space which are neither parallel nor intersecting; they are non-coplanar. Used In: Problem Family F8.

[Shortest Distance between Skew Lines (Vector Form)] NCERT Reference: Section 11.5.1 Formula: $d = |(b_1 \times b_2) \cdot (a_2 - a_1)| / |b_1 \times b_2|$ Used In: Problem Family F8.

[Shortest Distance between Parallel Lines (Vector Form)] NCERT Reference: Section 11.5.2 Formula: $d = |b \times (a_2 - a_1)| / |b|$ Note: In parallel lines, direction vectors are proportional, so we use a single vector b. Used In: Problem Family F9.

[Shortest Distance (Cartesian Form)] NCERT Reference: Section 11.5.1 Formula: d = | Numerator | / $\sqrt{((b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2)}$ Where the Numerator is the Determinant: | x_2-x_1 y_2-y_1 z_2-z_1 | | a_1 b_1 c_1 | | a_2 b_2 c_2 | Used In: Problem Family F8 (Cartesian variant).

Transitioning from these theoretical definitions, we must strictly adhere to the formal language and notations used in the NCERT syllabus to ensure maximum scoring potential.

SECTION 2: WHAT NCERT SAYS

To achieve full marks in CBSE board exams, precision is everything. NCERT provides a specific algorithmic path for these problems. Deviating from standard notations or omitting the vector setup can result in the loss of "step-marks," even if the final numerical answer is correct.

2.1 Key Statements According to the NCERT guidelines, students must internalize these properties:

1. If the shortest distance between two lines is exactly zero, the lines intersect at a point.
2. For skew lines, the line representing the shortest distance is perpendicular to both given lines.
3. Parallel lines are always coplanar; skew lines are always non-coplanar.
4. If direction ratios are proportional ($a_1/a_2 = b_1/b_2 = c_1/c_2$), the lines are parallel.
5. The shortest distance between parallel lines is the length of the perpendicular drawn from any point on one line to the other.

2.2 Examples and Exercises Study these benchmarks carefully, as they mirror the difficulty level of Board Exam questions:

- **Example 9 (Page 388):** Application of the Skew Line formula in Vector form. This is the "gold standard" for 4-mark questions.
- **Example 10 (Page 388):** Demonstrates the Parallel line formula. It teaches you to identify parallelism by checking direction vector proportionality.
- **Exercise 11.2 (Q12–Q13):** Standard practice for formula application.
- **Exercise 11.2 (Q14–Q15):** High-priority problems that often require converting Cartesian equations into Vector form before solving.

Successful problem-solving requires moving from static formula memorization to the active deployment of these strategies under exam conditions.

SECTION 3: PROBLEM-SOLVING AND MEMORY

Shortest Distance problems are highly algorithmic. Success does not require a complex derivation during the exam; it requires recognizing the "Problem Family" and following the established blueprint.

3.1 Problem Types

Problem Type [Family F8: Skew Line Distance]

- **Structural Goal:** Finding the gap between lines in different planes and directions.
- **Recognition Cues:** Direction ratios are NOT proportional (e.g., b_1 = 2i + 3j + 6k and b_2 = 3i - 5j + 2k).
- **Logic:** Finding the common perpendicular and projecting the point-difference onto it.
- **Pro-Tip:** If d = 0, you have proved the lines intersect.

Problem Type [Family F9: Parallel Line Distance]

- **Structural Goal:** Finding the fixed perpendicular gap between lines headed in the same direction.
- **Recognition Cues:** Direction vectors are identical or multiples of each other (e.g., b_1 = 2i + 3j + 6k and b_2 = 4i + 6j + 12k).
- **Logic:** Measuring the perpendicular height of one line from a point on the other.

Decision Table: Which Formula to Use? | Condition | Relationship | Problem Family | Formula Vector Key | | :--- | :--- | :--- | :--- | | $a_1/a_2 \neq b_1/b_2$ | Skew Lines | Family F8 | Use ($b_1 \times b_2$) | | $a_1/a_2 = b_1/b_2$ | Parallel Lines | Family F9 | Use b × (a_2 - a_1) | | d = 0 | Intersecting | Special Case | Shortest Distance is 0 |

3.2 Step-by-Step Methods

Blueprint for Family F8: Shortest Distance between Skew Lines

Pre-Check: Always check if the direction ratios are proportional. If they are not, proceed.

- **Step 1: Setup** Extract and list components explicitly (this is worth 1 mark): Line 1: $r = a_1 + \lambda b_1$ Line 2: $r = a_2 + \mu b_2$ Identify vectors a_1, b_1, a_2, and b_2.
- **Step 2: Transform** Compute the cross product $(b_1 \times b_2)$ using the **Determinant Method**. Then, calculate the magnitude $|b_1 \times b_2| = \sqrt{(x^2 + y^2 + z^2)}$.
- **Step 3: Apply** Find the difference vector $(a_2 - a_1)$. Compute the dot product: $(b_1 \times b_2) \cdot (a_2 - a_1)$. Substitute into the formula: $d = |(b_1 \times b_2) \cdot (a_2 - a_1)| / |b_1 \times b_2|$.
- **Step 4: Conclude** Ensure the distance is positive. State the final answer with "units."

3.3 How to Write Answers (The CBSE Way)

Answer Template: L1: "Let the given lines be $r = a_1 + \lambda b_1$ and $r = a_2 + \mu b_2$." L2: "Extracting the vectors, we have: $a_1 = \dots$, $b_1 = \dots$ $a_2 = \dots$, $b_2 = \dots$" (Crucial for method marks!) L3: "The shortest distance d is given by $d = |(b_1 \times b_2) \cdot (a_2 - a_1)| / |b_1 \times b_2|$." L4: "Calculating $b_1 \times b_2 = \dots$" L5: "Substituting values: d = |(Numerator Value)| / (Denominator Value)." L6: "Result = [Value] units."

Essential Phrases:

- "Comparing given equations with standard form $r = a + \lambda b$..."
- "Since $d \neq 0$, the given lines are skew."

3.4 Common Mistakes

Pitfall [1]: Negative Distance

- Wrong: Ending with $-5/\sqrt{26}$.
- ✓ Fix: Apply the absolute value bars. Distance is a magnitude; it cannot be negative.

Pitfall [2]: Using Skew Formula for Parallel Lines

- Wrong: Plugging parallel vectors into the Skew formula results in a zero denominator (division by zero).
- ✓ Fix: If b_1 and b_2 are proportional, use $d = |b \times (a_2 - a_1)| / |b|$.

Pitfall [3]: Determinant Sign Flip

- Wrong: Forgetting the negative sign for the middle (j-component) in the cross product.
- ✓ Fix: Use brackets: i(brackets) - j(brackets) + k(brackets).

3.5 Exam Strategy Master the **Vector form** first. It is the NCERT standard and is less prone to arithmetic errors than the long Cartesian formula. Even if the question provides Cartesian equations, quickly convert them to Vector form ($r = a + \lambda b$) to make your calculations cleaner and faster.

3.6 Topic Connections

- **Prerequisites:** Relies on Chapter 10 (Vectors) for Dot and Cross products.
- **Forward Links:** This "perpendicular distance" logic is the basis for the next sub-unit: "Distance of a point from a plane."

3.7 Revision Summary

- **Skew Lines** are non-parallel and non-intersecting.
- **Parallel Lines** have proportional direction ratios ($a_1/a_2 = b_1/b_2 = c_1/c_2$).
- **Shortest Distance** is the length of the common perpendicular.
- **Intersection:** If $d = 0$, the lines meet.
- **Method Marks:** Always extract a_1, b_1, a_2, b_2 and state the formula before calculating.
- **Units:** Never forget to write the word "units" after your final answer.
- **Memory Aid:** Remember **"Cross-Dot-Mag"** — **Cross** the directions, **Dot** with the point difference, divide by the **Magnitude** of the cross.

*** 💪 Master the extraction, check your ratios, and the marks are yours!***