CONCEPT QUICKSTART – Composition of Functions and Invertible Function

Unit: Unit1: Relations and Functions

Subject: For CBSE Class 12 Mathematics

SECTION 1: UNDERSTANDING THE CONCEPT

1.1 What Is This Concept?

In mathematics, after understanding what functions are—rules that map inputs from a domain to unique outputs in a co-domain—we can explore how to operate with them. This topic introduces two powerful operations: **composition** and **inversion**. At its core, function composition is the process of combining two functions to create a new one by applying them in sequence. Inversion is the process of finding a function that "reverses" the mapping of the original, taking outputs back to their corresponding inputs.

A critical point of clarity is distinguishing an inverse function from a reciprocal. The notation for an inverse function, $f^{-1}(x)$, looks deceptively similar to an exponent, which leads to a common misunderstanding. However, $f^{-1}(x)$ does **not** mean $1/f(x)$. The inverse $f^{-1}(x)$ is a function that undoes the action of $f(x)$, whereas $1/f(x)$ is simply the algebraic reciprocal of the function's output value. Mastering these two distinct concepts is fundamental to advanced mathematics.

1.2 Why It Matters

Understanding function composition and inversion is not just an academic exercise; it is a critical building block for higher-level mathematics. The concept of applying functions in a sequence directly lays the groundwork for the **Chain Rule** in calculus, one of the most essential rules for differentiation. Furthermore, the ability to find an inverse function is fundamental to solving complex equations and reversing mathematical processes. It allows us to isolate variables and determine the original input that produced a given output, a process central to fields ranging from cryptography to engineering.

1.3 Prior Learning Connection

To fully grasp composition and invertibility, you must be confident with the following prerequisite concepts from Class 11. These ideas form the foundation upon which the new concepts are built.

- **Definition of a Function:** A function is a special type of relation where every input has exactly one output. This is the basic object we are working with.

- **Domain, Co-domain, and Range of a Function:** You must know what set of inputs a function is defined for (domain), what set of outputs it could possibly map to (co-domain), and what set of outputs it actually produces (range). This is crucial for checking if a composition is even possible.
- **Types of Functions (One-one and Onto):** Understanding whether a function is one-one (injective) or onto (surjective) is the absolute key to determining if it can be inverted.

1.4 Core Definitions

The following are the formal definitions and theorems from the NCERT textbook that define this topic.

- **Composition of Functions**
  - **NCERT Reference:** Page 12, Definition 8
  - **Definition:** Let f: A → B and g: B → C be two functions. Then the composition of f and g, denoted by g ∘ f, is defined as the function (g ∘ f): A → C given by (g ∘ f)(x) = g(f(x)), ∀ x ∈ A.
  - **Used In:** Composition Calculation & Property Check
- **Invertible Function**
  - **NCERT Reference:** Page 12, Definition 9
  - **Definition:** A function f: X → Y is defined to be invertible if there exists a function g: Y → X such that g ∘ f = I_X and f ∘ g = I_Y. The function g is called the inverse of f and is denoted by f^{-1}.
  - **Used In:** Invertibility & Inverse Finding
- **Condition for Invertibility**
  - **NCERT Reference:** Page 12, paragraph following Definition 9
  - **Definition:** A function f is invertible if and only if f is one-one and onto (bijective).
  - **Used In:** Proving Invertibility

SECTION 2: WHAT NCERT SAYS

2.1 Key Statements

This section distills the most critical statements from the NCERT textbook, providing a concise summary of the official curriculum's perspective. These are the core principles you are expected to understand and apply.

1. **Condition for Composition:** For the composite function $g \circ f$ to be defined, the output of the first function (f) must be a valid input for the second function (g). This means the range of f must be a subset of the domain of g.
2. **Composition is Not Commutative:** In general, the order in which you compose functions matters. Applying f then g ($g \circ f$) is usually not the same as applying g then f ($f \circ g$). As shown in the textbook, $g \circ f \neq f \circ g$ for most functions.
3. **Invertibility Requires Bijectivity:** A function has an inverse if and only if it is **bijective**, meaning it must be both **one-one** (every output comes from a unique input) and **onto** (every possible output is actually produced). This is the definitive test for invertibility.
4. **The Role of an Inverse:** The inverse function, f^{-1}, is defined by its ability to undo the original function. Composing a function with its inverse in either order results in the identity function, which simply returns the original input: $(f^{-1} \circ f)(x) = x$ and $(f \circ f^{-1})(y) = y$.

The following worked examples from the textbook demonstrate these principles in action.

2.2 Examples and Exercises

The NCERT textbook uses several key examples to illustrate the mechanics of function composition and inversion. Understanding these examples is crucial for building problem-solving skills.

- **Example 15, Page 12**
  - **What it Shows:** Demonstrates the direct computation of $g \circ f$ for functions with finite, explicitly defined domains and mappings.
  - **Why it's Important:** It builds foundational understanding of the composition process step-by-step, showing how the output of f becomes the input for g.
- **Example 16, Page 12**
  - **What it Shows:** Calculates both $g \circ f$ and $f \circ g$ for two continuous functions, $f(x) = \cos x$ and $g(x) = 3x^2$, and proves that they are not equal.
  - **Why it's Important:** It concretely establishes that function composition is not commutative, a key property to remember.
- **Example 17, Page 12**
  - **What it Shows:** A complete demonstration of how to prove a function is invertible and then find its inverse. It takes the function $f(x) = 4x + 3$ and derives its inverse $g(y) = (y-3)/4$.

- **Why it's Important:** This example serves as a model solution for the most common type of invertibility question, showing both the proof and the algebraic derivation.

Relevant Exercises

For practice, the following questions from the **Miscellaneous Exercise on Chapter 1** are directly relevant to this topic:

- Questions 1–7 (Note: These questions cover composition, invertibility, and related properties of functions from the chapter.)

SECTION 3: PROBLEM-SOLVING AND MEMORY

3.1 Problem Types

NCERT and board exam questions tend to follow predictable patterns. Recognizing these patterns, or "Problem Types," is the key to selecting the right method and solving questions efficiently. For this topic, there are two main families of problems.

- **Problem Type: Composition Calculation & Property Check**
  - **Structural Goal:** To combine two given functions f and g into a new composite function, or to compare different compositions.
  - **Recognition Cues:**
    - **Surface:** Keywords like "Find g ∘ f", "Find f ∘ g", "Show f ∘ g ≠ g ∘ f".
    - **Structural:** You are given two functions, f(x) and g(x), and are asked to combine them sequentially or compare the results of different combination orders.
  - **What You're Really Doing:** Substituting the entire expression for the inner function into the variable of the outer function.
  - **NCERT References:** Examples: 15, 16 | Exercises: Miscellaneous Exercise Qs
  - **Confusable Types:** Function addition (f+g)(x), which is simple addition, not nested evaluation.
- **Problem Type: Invertibility & Inverse Finding**
  - **Structural Goal:** To prove that a function has a valid inverse and then to derive the formula for that inverse.
  - **Recognition Cues:**

  - **Surface:** Phrases like "Show f is invertible", "Find the inverse of f", "Prove f is bijective".
  - **Structural:** You are given a single function f and are asked to prove it is both one-one and onto, and then derive its inverse, f^{-1}.
- **What You're Really Doing:** First, performing the two-part proof for bijectivity (one-one and onto). Second, performing algebraic manipulation to solve for the input variable in terms of the output variable.
- **NCERT References:** Examples: 17 | Exercises: Miscellaneous Exercise Qs
- **Confusable Types:** Finding the reciprocal 1/f(x), which is a simple algebraic operation, not the reversal of the function's mapping.

3.2 Step-by-Step Methods

For the most common and complex problem types, there is a reliable, step-by-step method that will lead to the correct solution if followed carefully.

- **Type: Invertibility & Inverse Finding: Solution Method**
  - **Pre-Check:** State the essential pre-condition: "To be invertible, a function must be bijective (both one-one and onto). This must be proven first."
  - **Core Steps:**
    1. **Step 1: Prove One-One (Setup):** Start with the test condition $f(x_1) = f(x_2)$. Perform the necessary algebraic steps to show that this condition implies $x_1 = x_2$.
    2. **Step 2: Prove Onto (Apply):** State the goal: for any element y in the co-domain, we must find an element x in the domain such that f(x) = y. Usually, this involves setting y = f(x) and showing that a valid x can be found.
    3. **Step 3: Conclude Invertibility (Verify):** After successfully completing the first two steps, state that since the function is proven to be both one-one and onto, it is invertible.
    4. **Step 4: Find the Inverse (Solve):** Take the equation y = f(x) that you used in the "onto" proof and algebraically solve for x in terms of y.
    5. **Step 5: State the Inverse Function (Conclude):** Write the final inverse function, typically as $f^{-1}(y)$ = [expression in y] or by replacing y with x to get $f^{-1}(x)$ = [expression in x].
  - **Variants:** No significant variants are noted in the source material for this core method.

- **When NOT to Use:** Do not use this method if the function is not bijective. If either the one-one test or the onto test fails, the function is not invertible, and the process stops there.

3.3 How to Write Answers

In board exams, presenting your solution in the expected format is just as important as getting the correct numerical answer. A clear, logical structure is crucial for earning full marks.

- **Answer Template:** Proving Invertibility Frame
- **When to Use:** When a question asks to "Show that f is invertible and find its inverse".
- **Line-by-Line:**
  - **L1 (Prove One-One):** Start with "Let x_1, x_2 be two elements in the domain such that $f(x_1) = f(x_2)$." Show the algebraic steps clearly and conclude with "...which implies $x_1 = x_2$. Therefore, f is one-one."
  - **L2 (Prove Onto):** Start with "Let y be an arbitrary element in the co-domain." Show the steps to find a corresponding x and conclude with "Thus, for every y in the co-domain, there exists an x in the domain such that f(x) = y. Therefore, f is onto."
  - **L3 (Conclude Invertibility):** State clearly, "Since f is both one-one and onto, it is a bijective function. Hence, f is invertible."
  - **L4 (Derive Inverse):** Write "To find the inverse, let y = f(x)." Show all the algebraic steps required to solve for x in terms of y.
  - **L5 (State Final Answer):** Conclude with the formal statement of the inverse: "Therefore, the inverse function f^{-1} is given by $f^{-1}(y) = ...$".
- **Essential Phrases:**
  - "$f(x_1) = f(x_2)$"
  - "For every $y \in Y$"
  - "Since f is bijective, it is invertible."
- **General Rules:**
  - Clearly define the domain and co-domain of the function at the start.
  - Show each logical step of the proof; do not skip algebraic manipulations.
  - Use correct mathematical notation (∀, ∃, ∈) where appropriate.
- **Type-Specific:** For invertibility proofs, the one-one and onto sections are mandatory components. They must be shown completely and cannot be assumed or skipped.

3.4 Common Mistakes

Learning to avoid common errors is one of the fastest ways to improve accuracy. Pay close attention to these frequent pitfalls.

Pitfalls to Avoid

- **Pitfall 1: Confusing Composition Order**
  - **Category:** Logical
  - **Occurs In:** Composition Calculation & Property Check
  - **Wrong:** Assuming $(g \circ f)(x)$ is the same as $(f \circ g)(x)$.
  - ✓ **Fix:** Always apply the inner function first: $g \circ f$ means $g(f(x))$. Remember that function composition is generally not commutative.
- **Pitfall 2: Confusing Inverse with Reciprocal**
  - **Category:** Logical/Notation
  - **Occurs In:** Invertibility & Inverse Finding
  - **Wrong:** Calculating $1/f(x)$ when asked for $f^{-1}(x)$.
  - ✓ **Fix:** Remember that $f^{-1}(x)$ is the function that reverses the mapping of f, found by solving $y=f(x)$ for x. It is not the algebraic reciprocal.
- **Pitfall 3: Algebraic Errors in Solving for Inverse**
  - **Category:** Algebra
  - **Occurs In:** Invertibility & Inverse Finding, Step 4 (Solve)
  - **Wrong:** Making sign errors or incorrect manipulations when isolating x from the equation $y = f(x)$.
  - ✓ **Fix:** Perform the algebraic steps carefully, one by one. After finding the inverse, you can quickly check your answer by computing $f(f^{-1}(x))$ to see if it simplifies back to x.

Critical Conditions to Check

- **Condition 1: Domain-Range Compatibility for Composition**
  - **Rule:** For the composition $g \circ f$ to be defined, the range of the first function (f) must be a subset of the domain of the second function (g).
  - **When:** Before attempting to calculate $(g \circ f)(x)$.
  - **Linked:** Problem Type: Composition Calculation & Property Check.

3.5 Exam Strategy

A strategic approach to preparation will help you focus your efforts on the most important areas.

- **Example Range:** Master the key NCERT examples for this topic: **Examples 15, 16, and 17 on Pages 12-13.**
- **Exercise Sets:** The primary set for practice is the **Miscellaneous Exercise on Chapter 1.**
- **Question Patterns:** Expect to see these three common question types:
    1. Compute $g \circ f$ and $f \circ g$ for given functions f and g.
    2. Show that a function is invertible by proving it is both one-one and onto.
    3. Find the explicit formula for the inverse function f^{-1}.
- **Approach:** Recommend a learning sequence: Master the mechanical process of **composition** first (Examples 15, 16). Then, focus on the complete **invertibility proof** structure (Example 17), as it combines multiple concepts and is a common high-value question.

3.6 Topic Connections

Mathematical concepts are rarely isolated. Understanding how they connect to past and future topics deepens your comprehension.

- **Prerequisites:**
    - **Definition of a Function:** This is the core object we are composing and inverting. A clear understanding of what a function is, is non-negotiable.
    - **Domain, Co-domain, and Range of a Function:** These concepts are essential for determining if a composition $g \circ f$ is valid (Range of $f \subseteq$ Domain of g).
    - **Types of Functions (One-one and Onto):** This is not just a prerequisite; it is the fundamental condition required to prove a function is invertible. Without mastery of these proofs, you cannot solve invertibility problems.
- **Forward Links:**
    - **Calculus: Chain Rule:** The process of differentiating a composite function, $(g \circ f)'(x)$, is a direct application of the idea of nested functions. The chain rule is essentially the calculus equivalent of function composition.

3.7 Revision Summary

This section provides a high-density summary for quick last-minute revision.

- **Key Points:**
    1. The composition of two functions $f: A \to B$ and $g: B \to C$ is the function $g \circ f: A \to C$ defined by $(g \circ f)(x) = g(f(x))$ for every $x \in A$.
    2. A function $f: X \to Y$ is invertible if there exists a function $g: Y \to X$ such that $g \circ f = I_X$ and $f \circ g = I_Y$, where I_X and I_Y are identity functions. The function g is the inverse of f, denoted f^{-1}.
    3. A function f is invertible if and only if it is both one-one and onto (bijective). This is the key condition for proving invertibility.
- **Memory Aids:**
    - No specific memory aids like mnemonics are provided in the source documents.