CONCEPT QUICKSTART – Types of Functions

Unit: Unit1: Relations and Functions

Subject: For CBSE Class 12 Mathematics

SECTION 1: UNDERSTANDING THE CONCEPT

1.1 What Is This Concept?

In mathematics, a function is a fundamental tool for describing relationships between quantities. While all functions follow the basic rule of mapping inputs from a domain to outputs in a co-domain, the *way* they perform this mapping can vary significantly. Some functions assign a unique output to every unique input, while others might map several different inputs to the same output. Similarly, some functions utilize every possible output in their co-domain, while others use only a fraction of them. These structural differences in mapping are not trivial; they give rise to essential classifications that define a function's behavior and capabilities.

The Big Idea: The core concept is classifying functions based on two key properties: whether each input maps to a unique output (**one-one** or *injective*), and whether every possible output is actually used (**onto** or *surjective*).

These classifications are more than just labels; they are crucial determinants of a function's fundamental properties. For instance, knowing if a function is both one-one and onto tells us whether it is possible to define a reverse function that can perfectly undo the original mapping. This property, known as invertibility, is central to many areas of advanced mathematics.

A common point of confusion is thinking that these properties are dependent on each other. It is critical to understand that **"one-one" and "onto" are two independent characteristics**. A function can be one-one but not onto, onto but not one-one, both, or neither. Each of these four possibilities describes a distinct type of mapping behavior. Understanding this classification system is the first step toward a deeper mastery of functions and their applications.

1.2 Why It Matters

Understanding the different types of functions is not merely an abstract classification exercise; it is a strategic necessity for building a strong foundation in higher mathematics. The properties of being one-one and onto are fundamental concepts that unlock more advanced topics and are, as the NCERT textbook notes, of "paramount importance in mathematics and among other disciplines".

The most immediate and critical application of this concept is in the study of **Invertible Functions**. A function can only be reversed, or have an inverse, if and only if it is **bijective**—that is, it must be both one-one and onto. This condition ensures that for any output, we can trace it back to a single, unique input, which is the very definition of an inverse function. This idea is not just confined to this chapter; it has far-reaching implications in later topics, including:

- **Inverse Trigonometric Functions:** Defining functions like $\sin^{-1}(x)$ requires restricting the domain of sin(x) to an interval where it is bijective.
- **Calculus:** Concepts like the substitution rule in integration and finding derivatives of inverse functions rely on a solid understanding of these properties.

Grasping these classifications is therefore essential, as they form the logical underpinning for many subsequent mathematical concepts.

1.3 Prior Learning Connection

The concepts covered in this topic build directly upon the foundational knowledge of functions you acquired in Class 11. To successfully classify functions, you must be comfortable with the following prerequisite concepts:

- **The notion of a function:** You must fundamentally understand what a function is—a special type of relation that assigns exactly one output to each input—before you can begin to classify its different types.
- **Domain, Co-domain, and Range:** These three sets are the building blocks for analyzing functions. The concepts of domain and co-domain are essential for defining the function's scope, while the range is crucial for testing the "onto" property, which involves comparing the range directly to the co-domain.
- **Graphs of standard functions:** Visualizing a function's graph provides a quick, intuitive check. For instance, the **Horizontal Line Test** (where a function is one-one if no horizontal line intersects its graph more than once) is a powerful visual tool, even though a formal algebraic proof is required for exams.

This chapter will take these Class 11 concepts and apply a more rigorous, proof-based layer of analysis, leading to the formal definitions and methods used in Class 12.

1.4 Core Definitions

The NCERT textbook provides precise, formal definitions for classifying functions. These definitions form the logical basis for all proofs and problem-solving techniques related to this topic.

- **One-one function**
  - **NCERT Reference:** Definition 5, Page 7

- **Definition:** A function f: X → Y is defined to be one-one (or injective), if the images of distinct elements of X under f are distinct, i.e., for every $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Otherwise, f is called many-one.
- **Used In:** OO (One-One Check), BJ (Bijective Verification)

- **Onto function**
    - **NCERT Reference:** Definition 6, Page 7
    - **Definition:** A function f: X → Y is said to be onto (or surjective), if every element of Y is the image of some element of X under f, i.e., for every y ∈ Y, there exists an element x in X such that f(x) = y.
    - **Used In:** ONT (Onto Check), BJ (Bijective Verification)
- **One-one and Onto function**
    - **NCERT Reference:** Definition 7, Page 8; its link to invertibility is formalized in Definition 9, Page 12.
    - **Definition:** A function f: X → Y is said to be one-one and onto (or bijective), if f is both one-one and onto.
    - *Note: This property of bijectivity is the direct and non-negotiable prerequisite for a function to be invertible, as discussed in "Why It Matters."*
    - **Used In:** BJ (Bijective Verification), INV (Invertibility & Inverse Finding)

These definitions are the authoritative rules of the game. Every problem you solve will require you to demonstrate that a function either satisfies or fails to satisfy these precise conditions.

SECTION 2: WHAT NCERT SAYS

2.1 Key Statements from the Textbook

This section distills the most important principles regarding types of functions directly from the NCERT chapter. These statements go beyond the basic definitions and provide the core logic used in proofs and problem-solving.

1. **The Test for Injectivity (One-One):** To formally prove that a function f is one-one, you must start by assuming that $f(x_1) = f(x_2)$ for any two arbitrary elements x_1 and x_2 in the domain. Through logical and algebraic steps, you must demonstrate that this assumption necessarily leads to the conclusion that $x_1 = x_2$.

2. **The Test for Surjectivity (Onto):** A function f: X → Y is onto if and only if its **Range is equal to its Co-domain (Y)**. Operationally, this means that for any arbitrarily chosen element y from the co-domain Y, you must be able to find at least one element x in the domain X such that f(x) = y.
3. **The Bijectivity Condition:** A function is classified as bijective if and only if it successfully passes *both* the test for being one-one and the test for being onto. Failing even one of these tests means the function is not bijective.
4. **The Finite Set Property:** A special property exists for functions between two *finite sets that have the same number of elements*. In such cases, if the function is proven to be one-one, it is automatically guaranteed to be onto. Conversely, if it is proven to be onto, it is guaranteed to be one-one.
5. **The Infinite Set Distinction:** The special property for finite sets **does not apply** to functions defined on infinite sets (like the set of Natural Numbers, **N**, or Real Numbers, **R**). For infinite sets, a function can be one-one without being onto, or onto without being one-one. These properties must be checked independently.

These principles form the strategic foundation for approaching any question on the topic.

2.2 Examples and Exercises

The NCERT textbook provides several key examples that illustrate how to apply the core definitions and principles to solve common problems. Understanding these examples is crucial for mastering the required techniques.

- **Example 8 (Page 8)**
  - **Function:** f: N → N, defined by f(x) = 2x.
  - **What it shows:** How a function can be **one-one but not onto**.
  - **Why it's important:** This is a classic demonstration that for infinite sets, injectivity does not imply surjectivity. The function is one-one because $2x_1 = 2x_2$ implies $x_1 = x_2$. However, it is not onto because the range consists only of even natural numbers, which is a proper subset of the co-domain **N** (e.g., the odd number 1 has no pre-image).
  - **Curriculum Insight:** NCERT uses this simple function to force students to confront the distinction between finite and infinite sets. The result feels counter-intuitive, which is a deliberate technique to build a more robust, formal understanding.
- **Example 11 (Page 9)**
  - **Function:** f: R → R, defined by $f(x) = x^2$.

- What it shows: How a function can be **neither one-one nor onto**.

  - **Why it's important:** It illustrates two common ways a function can fail the tests. It fails the one-one test because different inputs can produce the same output (e.g., f(-1) = 1 and f(1) = 1). It fails the onto test because the range includes only non-negative real numbers, meaning no negative number in the co-domain **R** has a pre-image.
  - **Curriculum Insight:** This is the canonical example for a function that is neither injective nor surjective. It is chosen because the algebraic reasons (f(-x) = f(x)) and the graphical reasons (a parabola fails the horizontal line test) are easily accessible to students.

- **Example 12 (Page 9)**
  - **Function:** f: N → N, a piecewise function where f(x) = x + 1 if x is odd, and f(x) = x - 1 if x is even.
  - **What it shows:** A rigorous proof that a more complex function is **both one-one and onto (bijective)**.
  - **Why it's important:** It demonstrates the critical technique of proving by cases. To prove injectivity and surjectivity, the inputs must be considered separately for odd and even values, showing how to handle different scenarios within a single proof.
  - **Curriculum Insight:** This example is designed to move beyond simple algebraic functions and test a student's ability to apply definitions using case-based logical reasoning, a critical skill in advanced mathematics.

Core Practice Problems

To build proficiency, focus on the following exercises from the textbook:

- **Exercise 1.2, Questions 1–12.**

These problems provide comprehensive practice across all the major problem types and variations you are likely to encounter.

SECTION 3: PROBLEM-SOLVING AND MEMORY

3.1 Problem Types

Nearly all examination questions about function types can be categorized into a few standard patterns. Recognizing these patterns is the key to knowing exactly what you need to prove and how to structure your solution.

Problem Type 1: Checking for One-one (Injectivity)

- **Structural Goal:** To prove that for any two inputs x_1 and x_2 in the domain, if their outputs $f(x_1)$ and $f(x_2)$ are the same, then the inputs themselves must also have been the same ($x_1 = x_2$).
- **Recognition Cues:**
  - **Surface:** "Show that f is one-one", "Check for injectivity", "Is f injective?"
  - **Structural:** You are given a function f(x) with a defined domain and co-domain and are asked to test this specific property.
- **What You're Really Doing:** Algebraically proving that it is impossible for two different inputs to ever lead to the same output.
- **NCERT References:** Examples 7, 8, 9, 11 | Exercises 1.2 (Q1, 2, 7)
- **Confusable Types:** Checking for Onto (Surjectivity).

Problem Type 2: Checking for Onto (Surjectivity)

- **Structural Goal:** To prove that for any element y chosen from the co-domain, you can always find an element x in the domain that maps to it, i.e., f(x) = y.
- **Recognition Cues:**
  - **Surface:** "Show that f is onto", "Check for surjectivity", "Is f surjective?"
  - **Structural:** You are given a function f(x) and must verify if its range covers the entire specified co-domain.
- **What You're Really Doing:** Showing that the function doesn't "miss" any of the possible output values listed in its co-domain.
- **NCERT References:** Examples 9, 10 | Exercises 1.2 (Q1-3)
- **Confusable Types:** Checking for One-one (Injectivity).

Problem Type 3: Checking for Bijectivity

- **Structural Goal:** To successfully perform *both* the one-one check and the onto check for the given function.
- **Recognition Cues:**

  - **Surface:** "Show that f is bijective", "Prove the function is one-one and onto", "Is f invertible?"
  - **Structural:** This question requires a two-part proof. You must first prove injectivity and then separately prove surjectivity.
- **What You're Really Doing:** Verifying that the mapping is a perfect, one-to-one correspondence between the elements of the domain and the co-domain.
- **NCERT References:** Examples 12, 13, 14 | Exercises 1.2 (Q2, Q8-12)
- **Confusable Types:** Proving only one of the two required properties.

Once you identify the problem type, you can confidently apply the corresponding step-by-step method to construct your solution.

3.2 Step-by-Step Methods

Here are the procedural blueprints for solving the primary problem types. Following these steps methodically will ensure a complete and logical proof.

Method 1: Checking for One-one (Injectivity)

- **Pre-Check:** Clearly identify the function's **Domain (X)** and **Co-domain (Y)** from its definition f: X → Y. This context is crucial.
- **Core Steps:**
  1. **Step 1 (Setup):** Begin your proof by assuming $f(x_1) = f(x_2)$ for two arbitrary elements x_1, x_2 from the domain X.
  2. **Step 2 (Apply):** Substitute the specific rule for f(x) into your assumption. For example, if f(x) = 3x - 4, your equation becomes $3x_1 - 4 = 3x_2 - 4$.
  3. **Step 3 (Solve):** Use standard algebraic manipulation (e.g., adding 4 to both sides, dividing by 3) to simplify the equation and isolate x_1.
  4. **Step 4 (Conclude):**
     - If your algebra simplifies perfectly to $x_1 = x_2$, you have proven the function is **one-one**.
     - If you cannot prove $x_1 = x_2$ generally, find a concrete **counterexample** (e.g., for $f(x) = x^2$, show that f(-1) = f(1) but -1 ≠ 1) to prove the function is **not one-one**.
- **Variants:** For piecewise functions, you may need to check different cases (e.g., both x_1, x_2 are even; one is even, one is odd).

- **When NOT to Use:** While a graphical method like the Horizontal Line Test can give you a quick intuition, it is not a substitute for the formal algebraic proof required in exams.

Method 2: Checking for Onto (Surjectivity)

- **Pre-Check:** Clearly identify the **Co-domain (Y)**. This is the target set of outputs that your function's range must completely cover.
- **Core Steps:**
  1. **Step 1 (Setup):** Let y be an arbitrary element from the co-domain Y. Set up the equation y = f(x). Your goal is to see if you can always find an x that satisfies this.
  2. **Step 2 (Solve for x):** Use algebraic manipulation to rearrange the equation and express x in terms of y. For example, if y = 2x + 3, solving for x gives x = (y - 3) / 2.
  3. **Step 3 (Verify):** This is the most critical step. For any y chosen from the co-domain, you must check if the value of x you found in Step 2 is a valid element of the function's **domain**. This step is where students most often make the critical error of **Ignoring the Domain and Co-domain** (see Pitfall 3). Always perform this check meticulously.
  4. **Step 4 (Conclude):**
     - If the x you found is *always* in the domain for *any* y from the co-domain, the function is **onto**.
     - If you can find even one value of y in the co-domain for which the corresponding x is *not* in the domain, you have found a counterexample, and the function is **not onto**.

Knowing these steps is essential, but presenting your argument clearly is equally important for securing full marks.

3.3 How to Write Answers

In CBSE board exams, presenting your answer in a clear, structured, and logical format is crucial. Simply getting the right conclusion is not enough; the reasoning must be explicitly shown. Use the following template for writing formal proofs.

- **Answer Template: Proof of Injectivity (One-one)**
- **When to Use:** When a question asks you to "Show that f is one-one" or "Check for injectivity."
- **Line-by-Line:**
  - **L1 (State Function):** State the function f: X → Y and its rule, f(x).

- L2 (Setup Assumption): "Let $x_1, x_2 \in X$ be two arbitrary elements such that $f(x_1) = f(x_2)$."
- **L3 (Substitute Rule):** Substitute the function's rule into the equation.
- **L4 (Solve Algebraically):** Show the simplification steps.
- **L5 (Conclude Formally):** "Since $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, the function f is one-one (injective)."

- **Answer Template: Proof of Non-Injectivity (by Counterexample)**
- **When to Use:** When asked to show a function is "not one-one" or "many-one."
- **Line-by-Line:**
  - **L1 (State Function):** State the function $f: X \to Y$ and its rule, $f(x)$.
  - **L2 (Identify Counterexample):** "Consider two distinct elements in the domain, for instance, $x_1 = -1$ and $x_2 = 1$ in R."
  - **L3 (Show Outputs are Equal):** Calculate $f(x_1)$ and $f(x_2)$. "$f(-1) = (-1)^2 = 1$" and "$f(1) = (1)^2 = 1$."
  - **L4 (State the Contradiction):** "Here, $f(x_1) = f(x_2)$ but $x_1 \neq x_2$."
  - **L5 (Conclude Formally):** "Therefore, since distinct elements in the domain map to the same element in the co-domain, the function f is not one-one (it is many-one)."
- **Essential Phrases:**
  - "Let $x_1, x_2 \in$ [Domain Name]..."
  - "...such that $f(x_1) = f(x_2)$."
  - "Therefore, f is one-one."
- **General Rules for All Proofs:**

1. Always define your variables and the set they belong to (e.g., "Let $y \in R$ be an arbitrary element").

2. Show each logical step clearly. Use the implies symbol ($\Rightarrow$) to connect steps in a logical chain.

3. State your final conclusion clearly and unambiguously.

4. **Important:** If you are proving a function is **NOT** one-one or **NOT** onto, a single, clear counterexample is both sufficient and the preferred method.

Adhering to this structure ensures your answer is not only correct but also easy for an examiner to follow and grade.

3.4 Common Mistakes to Avoid

Even with a solid understanding of the methods, students can lose marks due to common conceptual and procedural errors. Being aware of these pitfalls is the first step to avoiding them.

- **Pitfall 1: Confusing One-one and Onto**
  - **Category:** Logical
  - **Occurs In:** All problem types, especially during the final conclusion.
  - **Wrong:** Proving a function is one-one and then incorrectly assuming it must also be onto (or vice-versa). This is a very common error when dealing with infinite sets.
  - ✓ **Fix:** Always remember that injectivity and surjectivity are **separate properties**. They must be checked independently unless the question involves a function between two *finite sets of the same size*.
- **Pitfall 2: Testing with Examples Instead of Proving Generally**
  - **Category:** Logical
  - **Occurs In:** One-one and Onto checks, during the main proof.
  - **Wrong:** Showing f(1) = 2 and f(2) = 4 for f(x) = 2x and concluding that it is one-one. This only shows it's true for those two numbers, not for all numbers.
  - ✓ **Fix:** To prove a property is **TRUE**, you must use arbitrary variables (like x_1, x_2, or y) to show it holds for all cases. Use specific numbers only to prove a property is **FALSE** (i.e., to provide a counterexample).
- **Pitfall 3: Ignoring the Domain and Co-domain**
  - **Category:** Logic
  - **Occurs In:** The "Verify" step (Step 3) of an Onto check.
  - **Wrong:** For a function f: N → N given by y = 2x, solving for x gives x = y/2. A student might stop here and conclude it's onto. They forget to check if y/2 is always a natural number. (It is not; if y=3, then x=1.5, which is not in the domain **N**).
  - ✓ **Fix:** After solving for x in terms of y in an onto proof, **always ask the critical question:** "For any y I pick from the co-domain, is this resulting value of x guaranteed to be in the domain?" If the answer is no, the function is not onto.

3.5 Exam Strategy

To master this topic for exams, focus on pattern recognition, targeted practice, and a systematic study approach.

- **Key NCERT Resources for Practice:**
  - **Example Range:** Work through and fully understand **Examples 7 through 14** (Pages 8-10). These cover the most important scenarios.
  - **Exercise Sets:** Your goal should be to master all questions in **Exercise 1.2 (Questions 1-12)**.
- **Common Question Patterns to Master:**

1. Check injectivity and surjectivity for a standard algebraic function (e.g., linear, quadratic, cubic).

2. Analyze how the properties of a single function rule (like $f(x) = x^2$) change when the domain and co-domain are switched (e.g., from $f: N \to N$ to $f: R \to R$). (This is a high-yield pattern for examiners as it directly tests conceptual understanding over rote memorization. Master this variation.)

3. Justify whether a given function is one-one, onto, or bijective, and state a clear conclusion.

4. Prove properties for piecewise-defined functions, requiring you to construct proofs by cases (e.g., odd/even inputs), as seen in Example 12.

- **Recommended Study Approach:**

1. **Memorize the Definitions:** First, ensure you can write the formal definitions of one-one, onto, and bijective from memory.

2. **Practice the Methods:** Practice the step-by-step algebraic proof methods for functions on the number sets **N**, **Z**, and **R**.

3. **Focus on Variations:** Pay special attention to how changing the domain and co-domain in the exercises can completely change whether a function is one-one or onto. This demonstrates a true understanding of the concepts.

This topic serves as a crucial gateway to the rest of the unit, particularly to the concept of invertible functions.

3.6 Topic Connections

The classification of functions is not an isolated topic; it is deeply interconnected with concepts that come before and after it in the curriculum.

- **Prerequisites (Looking Back):**

- **Class XI Functions:** The entire topic is a direct and more rigorous extension of the definition of a function, domain, co-domain, and range learned in Class XI.
- **Set Theory:** You cannot perform the check for an 'onto' function without a clear understanding that the co-domain is a set, and you are checking if the range is a subset equal to it.

- **Forward Links (Looking Ahead):**
  - **Invertible Functions:** This is the most significant and immediate connection. A function has an inverse **if and only if it is bijective (both one-one and onto)**. Mastering the checks for bijectivity is therefore a non-negotiable prerequisite for understanding invertible functions.
  - **Calculus:** In calculus, properties like injectivity are critical for defining inverse trigonometric and logarithmic functions over specific intervals where their standard counterparts are well-behaved and have a unique inverse.

Understanding these connections helps place the topic in its proper context as a cornerstone of advanced mathematics.

3.7 Revision Summary

This section provides a consolidated checklist of the most critical points for quick revision before an exam.

1. A function f: X → Y is **one-one (injective)** if $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for all x_1, x_2 in X.
2. A function f: X → Y is **onto (surjective)** if for every element y in the co-domain Y, there exists some element x in the domain X such that f(x) = y.
3. A function is **bijective** if it is both one-one and onto.
4. To **prove** a function is one-one, use a general algebraic proof starting with $f(x_1) = f(x_2)$.
5. To prove a function is **NOT** one-one, find a single concrete counterexample where two different inputs give the same output.
6. The most direct way to prove a function is onto is to show that its **Range = Co-domain**.
7. To prove a function is **NOT** onto, find a single element in the co-domain that has no pre-image in the domain.
8. For a function between two **finite sets of the same size**, being one-one automatically implies it is onto, and being onto implies it is one-one. This shortcut is **not valid** for infinite sets.